A REMARK ON THE WELL-POSEDNESS OF A DEGENERATED ZAKHAROV SYSTEM

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Abstract. We extend the local well-posedness theory for the Cauchy problem associated to a degenerated Zakharov system. The new main ingredients are the derivation of Strichartz and maximal function norm estimates for the linear solution of a Schrödinger type equation with missing dispersion in one direction. The result here improves the one in [10].

1. **Introduction.** We consider the initial value problem associated to the degenerate Zakharov system

$$\begin{cases} i(\partial_t E + \partial_z E) + \Delta_\perp E = nE, & (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ \partial_t^2 n - \Delta_\perp n = \Delta_\perp(|E|^2), \\ E(\cdot, 0) = E_0(\cdot), \ \ n(\cdot, 0) = n_0(\cdot), \ \ \partial_t n(\cdot, 0) = n_1(\cdot), \end{cases} \tag{1}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$, E is a complex-valued function, and n is a real-valued function. The system (1) describes the laser propagation when the paraxial approximation is used and the effect of the dispersion of the group velocity is negligible ([12]).

We use the term degenerate in the sense that there is no dispersion in the z-direction for the system in (1) in contrast to the well known Zakharov system

$$\begin{cases} i\partial_t E + \Delta E = nE, & (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ \partial_t^2 n - \Delta n = \Delta(|E|^2), \end{cases} \tag{2}$$

which was introduced in [13] to describe the long wave Langmuir turbulence in a plasma.

Regarding the IVP (1), Colin and Colin in [2] posed the question of the well-posedness. A positive answer was given by Linares, Ponce and Saut in [10], showing the local well-posedness of the IVP (1) in a suitable Sobolev space. The results proved in [10] extended previous ones for the Zakharov system (2), where transversal dispersion is taken into account (see [11], [5] and references therein). However, the system (1) is quite different from the classical Zakharov system (2) since the Cauchy problem for the periodic data exhibits strong instabilities of the Hadamard type implying ill-posedness (see [3]).

Our goal here is to extend the local well-posedness for the IVP (1) to a larger functional space than that in [10].

Before describing our main result and the new ingredients used in its proof we proceed as in [10] to study this problem.

First the IVP (1) is reduced into the IVP associated to a single equation, that is,

$$\begin{cases} i(\partial_t E + \partial_z E) + \Delta_\perp E = nE, \quad (x,y,z) \in \mathbb{R}^3, \ t > 0, \\ E(x,y,z,0) = E_0(x,y,z), \end{cases} \tag{3}$$

where

$$n(t) = N'(t)n_0 + N(t)n_1 + \int_0^t N(t-t')\Delta_\perp(|E(t')|^2)dt',$$

with

$$N(t)f = (-\Delta_\perp)^{-1/2}\sin((-\Delta_\perp)^{1/2}t)f, \tag{4}$$

and

$$N'(t)f = \cos((-\Delta_\perp)^{1/2}t)f, \tag{5}$$

where $(-\Delta_\perp)^{1/2}f = ((\xi_1^2 + \xi_2^2)^{1/2}\hat{f})^\vee$.

Then it is considered the integral equivalent formulation of the IVP (3), that is,

$$\begin{aligned} E(t) =& \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(N'(t')n_0 + N(t')n_1)E(t')dt' \\ &+ \int_0^t \mathcal{E}(t-t')\Big(\int_0^{t'} N(t'-s)\Delta_\perp(|E(s)|^2)\,ds\Big)E(t')\,dt', \end{aligned} \tag{6}$$

where $\mathcal{E}(t)$ denotes the unitary group associated to the linear problem to (3) given by

$$\mathcal{E}(t)E_0 = \big(e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)}\widehat{E_0}(\xi_1,\xi_2,\xi_3)\big)^\vee.$$

A smoothing effect for the unitary group $\mathcal{E}(t)$ similar to the one obtained for solutions of the linear Schrödinger equation was proved in [10] (see Proposition 2.1 below). This was the main tool used there to establish local well-posedness via contraction principle in the following functional space

$$\widetilde{H}^{2j+1}(\mathbb{R}^3) = \{f \in H^{2j+1}(\mathbb{R}^3), D_x^{1/2}\partial^\alpha f, D_y^{1/2}\partial^\alpha f \in L^2(\mathbb{R}^3), |\alpha| \le 2j+1, j \in \mathbb{N}\}, \tag{7}$$

where $\alpha \in (\mathbb{Z}^+)^3$ is a multiindex, $D_x^{1/2}f = (|\xi_1|^{1/2}\hat{f})^\vee$ and $D_y^{1/2}f = (|\xi_2|^{1/2}\hat{f})^\vee$.

Roughly the result in [10] guarantees the local well-posedness in $\widetilde{H}^{2j+1}(\mathbb{R}^3)$, $j \ge 2$, for data $E_0 \in \widetilde{H}^{2j+1}(\mathbb{R}^3)$, $n_0 \in H^{2j}(\mathbb{R}^3)$ and $n_1 \in H^{2j-1}(\mathbb{R}^3)$ with $\partial_z n_1 \in H^{2j-1}(\mathbb{R}^3)$, where $H^s(\mathbb{R}^3)$ is the usual Sobolev space.

To improve the previous result obtained in [10] we derive two new estimates for solutions of the linear problem. The first one is the following Strichartz estimate,

$$\|\mathcal{E}(t)f\|_{L_t^q L_{xy}^p L_z^2} \le c\|f\|_{L_{xyz}^2},$$

where $2/q = 1 - 2/p$, $2 \le p < \infty$.

We can observe that the lack of dispersion in the z-direction is reflected in the estimate above. The proof uses the explicit Fourier transform of e^{itx^2} and the usual method to prove Strichartz estimates for the linear Schrödinger equation.

The second new estimate for solutions of the linear problem is the following maximal function estimate

$$\|\mathcal{E}(t)f\|_{L_x^2 L_{yzT}^\infty} \le c(T,s)\|f\|_{H^s(\mathbb{R}^3)}, \quad s > 3/2. \tag{8}$$

The argument to prove (8) follows the ideas in [6], where they obtained a L_x^4-maximal function estimates for solutions of the linear problem associated to the modified Kadomtsev-Petviashvili (KPI) equation.

Remark 1. It is not clear whether the estimate (8) is sharp. In Proposition 2.4 below we show that this estimate is false in $H^s(\mathbb{R}^3)$ for $s < 1$.

To state our result we shall slightly modify the space $\widetilde{H}^{2j+1}(\mathbb{R}^3)$ defined in (7). We define

$$\widetilde{H}^2(\mathbb{R}^3) = \{f \in H^2(\mathbb{R}^3), D_x^{1/2}\partial^\alpha f, \ D_y^{1/2}\partial^\alpha f \in L^2(\mathbb{R}^3), \ |\alpha| = 2\}.$$

With this notation, the main result here reads as:

Theorem 1.1. *For initial data (E_0, n_0, n_1) in $\widetilde{H}^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $\partial_z n_1 \in H^1(\mathbb{R}^3)$, there exist $T > 0$ and a unique solution E of the integral equation (6) such that*

$$E \in C([0,T] : \widetilde{H}^2(\mathbb{R}^3)), \tag{9}$$

$$\sum_{|\alpha|=2} \left(\|\partial_x \partial^\alpha E\|_{L_x^\infty L_{yzT}^2} + \|\partial_y \partial^\alpha E\|_{L_y^\infty L_{xzT}^2} \right) < \infty, \tag{10}$$

$$\|E\|_{L_x^2 L_{yzT}^\infty} + \|E\|_{L_y^2 L_{xzT}^\infty} < \infty, \tag{11}$$

and

$$X_T(E) < \infty \tag{12}$$

where $X_T(\cdot)$ is defined in (13) below.

Moreover, there exists a neighborhood V of $(E_0, n_0, n_1) \in \widetilde{H}^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ such that the map $\mathcal{F} : (E_0, n_0, n_1) \mapsto E(t)$ from V into the class defined by (9)-(12) is smooth.

One also has that

$$n \in C([0,T] : H^2(\mathbb{R}^3)).$$

Remark 2. Observe that in comparison with the result in [10] we could considerably weaken the regularity required to prove local well-posedness for the IVP (1). Notice also that it would be possible to lower the regularity a little further because the maximal function works well in $H^s(\mathbb{R}^3)$, $s > 3/2$.

Remark 3. The Strichartz estimates were essential in our analysis. It may be possible to use them in the Bourgain spaces framework to obtain better results (see for instance [1], [5], and references therein for the Zakharov system). Regarding global well-posedness, we do not know any conserved quantity that might be useful to extend globally the local results.

The plan of the paper is the following. In Section 2 we prove the new linear estimates commented above and recall some known ones established in [10]. Some useful lemmas will also be presented in this section. In Section 3 we establish estimates involving the nonlinear term that allow us to simplify the exposition of the proof of the main result. Finally our main result will be proved in Section 4.

Before leaving this section we introduce the notation used throughout the paper. We use standard notation in Partial Differential Equations. In addition we will use c to denote various constants that may change from line to line.

Let $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. For $f = f(x,t) \in \mathcal{S}(\mathbb{R}^4)$, $\widehat{f}$ will denote its Fourier transform in space, whereas $\widehat{f}^{(x_i x_l)}$, respectively $\widehat{f}^{(x_i)}$, will denote its

Fourier transform in the $x_i x_l$ and x_i variables, $i, l = 1, 2, 3$. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s, by

$$\widehat{J_x^s f} = (1 + |\xi|^2)^{s/2} \widehat{f} \quad \text{and} \quad \widehat{D_x^s f} = |\xi|^s \widehat{f}.$$

We also use the notation $J_{x_i x_l}^s$ and $J_{x_i}^s$ to denote the operators

$$\widehat{J_{x_i x_l}^s f} = (1 + |(\xi_i, \xi_l)|^2)^{s/2} \widehat{f} \quad \text{and} \quad \widehat{J_{x_i}^s f} = (1 + |\xi_i|^2)^{s/2} \widehat{f}, \quad i, l = 1, 2, 3.$$

We introduce the next notation to set together all the terms involving the Strichartz norms in our analysis.

$$X_T(f) := \sum_{|\alpha|=1} \left(\|J_z^{1/4+} D_x^{1/2} \partial^\alpha f\|_{L_{xyT}^4 L_z^2} + \|J_z^{3/8+} \partial^\alpha f\|_{L_T^{8/3} L_{xy}^8 L_z^2} + \|J_z^{1/2+} \partial^\alpha f\|_{L_{xyT}^4 L_z^2} \right)$$
$$+ \sum_{|\alpha| \le 1} \left(\|\partial_x \partial^\alpha f\|_{L_{xyT}^4 L_z^2} + \|\partial_y \partial^\alpha f\|_{L_{xyT}^4 L_z^2} \right). \tag{13}$$

2. **Linear estimates.** Consider the linear problem:

$$\begin{cases} \partial_t E + \partial_z E - i\Delta_\perp E = 0, & \forall (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ E(x, y, z, 0) = E_0(x, y, z). \end{cases} \tag{14}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$.

The solution of the linear IVP (14) is given by the unitary group $\mathcal{E}(t) : H^s \to H^s$ such that

$$E(t) = \mathcal{E}(t) E_0 = \left(e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)} \widehat{E_0}(\xi_1, \xi_2, \xi_3) \right)^\vee. \tag{15}$$

Proposition 2.1. *The solution of the linear problem* (14) *satisfies*

$$\|D_x^{1/2} \mathcal{E}(t) f\|_{L_x^\infty L_{yzT}^2} \le c\|f\|_{L_{xyz}^2}, \tag{16}$$

$$\|D_x^{1/2} \int_0^t \mathcal{E}(t - t') G(t') dt'\|_{L_T^\infty L_{xyz}^2} \le c\|G\|_{L_x^1 L_{yzT}^2}, \tag{17}$$

and

$$\|\partial_x \int_0^t \mathcal{E}(t - t') G(t') dt'\|_{L_x^\infty L_{yzT}^2} \le c\|G\|_{L_x^1 L_{yzT}^2}. \tag{18}$$

These estimates hold exchanging x and y. Here $D_x^{1/2} f = (2\pi|\xi_1|^{1/2} \hat{f})^\vee$.

Proof. We refer to [10] for a proof of this proposition. $\qquad \square$

Now we give the precise statement of the inequality (8) and its proof.

Proposition 2.2. *For $s > 3/2$, and $T > 0$ we have*

$$\|\mathcal{E}(t) E_0\|_{L_x^2 L_{yzT}^\infty} \le c(T, s)\|E_0\|_{H^s(\mathbb{R}^3)}. \tag{19}$$

The same estimate holds exchanging x and y.

The proof of Proposition 2.2 is a direct consequence of the next lemma, as we shall see later.

Lemma 2.3. *For every $T > 0$ and $k \geq 0$, there exist a constant $c(T) > 0$ and a positive function $H_{k,T}(\cdot)$ such that*

$$\int_0^{+\infty} H_{k,T}(y)d\alpha \leq c(T)2^{3k}, \tag{20}$$

and

$$|\int_{\mathbb{R}^3} e^{i(-t(\xi_1^2+\xi_2^2+\xi_3)+x\cdot\xi)}\prod_{i=1}^3 \psi_i(\xi_i)\,d\xi| \leq H_{k,T}(|x_1|), \tag{21}$$

for $|t| \leq T$ and $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$ where $\psi_i(\xi_i) = \psi(2^{k+1} - |\xi_i|)$, and ψ denotes a $C^\infty(\mathbb{R})$ function such that $\psi = 1$ for $x \geq 1$ and $\psi = 0$ for $x \leq 0$.

To prove this lemma we will employ the argument introduced in [4].

Proof. Denote by $J(x, y, z, t)$ the integral on the left-hand side in (21). We can rewrite $J(t, x, y, z)$ as:

$$J(x, y, z, t) = \prod_{i=1}^3 J_i(x_i, t),$$

where

$$J_i(x_i, t) = \int e^{i\varphi_i(\xi_i)}\psi_i(\xi_i)d\xi_i \quad \text{and} \quad \varphi_i(\xi_i) = (-t\xi_i^2 + x_i\xi_i), \quad i = 1, 2,$$

and

$$J_3 = \int e^{i(-t\xi_3 + z\xi_3)}\psi_3(\xi_3)d\xi_3,$$

we have $|J| \leq |J_1\|J_2\|J_3|$.

Next we consider the following three cases:

- For $|x_1| < 1$ we use the support of ψ_j, $j = 1, 2, 3$, and get $|J| \leq c\,2^{3k}$.
- For $|x_1| \geq \max\{1, 2^3 2^k t\}$. In this case $|x_1| \geq 4|\xi_1|t$ for ξ_1 in the support of ψ_1, and so $|\varphi_1'(\xi_1)| \geq |x_1|/2$. Using integration by parts twice we get:

$$J_1 = \int e^{i\varphi_1}\Big(\frac{1}{\varphi_1'}\Big(\frac{\psi_1}{\varphi_1'}\Big)'\Big)' d\xi_1.$$

Now by the support of ψ_1 and the inequalities $|\varphi_1'(\xi_1)| \geq |x_1|/2$ and $|x_1|^{-1} \leq 1$ we have:

$$|J_1| \leq c(T)\int_{\{|\xi_1| \leq 2^{k+1}\}} \frac{1}{|x_1|^2}\,d\xi_1 \leq c(T)2^k|x_1|^{-2}.$$

Then $|J| \leq 2^{3k}c(T)|x_1|^{-2}$, using the supports of ψ_2 and ψ_3.

- For $1 \leq |x_1| \leq 2^3 2^k|t|$. Observe that in this case $t \geq 2^{-k-3} > 0$ and $t^{-2} \leq c|x_1|^{-2}2^{2k}$. Since $|\varphi_1''(\xi_1)| = 2t > 0$, Van der Corput lemma (see [9] for instance) implies $|J_1| \leq ct^{-1/2}$. Similarly, we have $|J_2| \leq ct^{-1/2}$. Thus $|J| \leq ct^{-1}2^k \leq cTt^{-2}2^k \leq c2^{3k}|x_1|^{-2}$ by using the support of ψ_3.

Finally, we define

$$H_{k,T}(\rho) = \begin{cases} c2^{3k} & \text{for} \quad 0 \leq \rho < 1, \\ c(T)2^{3k}\rho^{-2} & \text{for} \quad 1 \leq \rho, \end{cases}$$

and this function satisfies (20) and (21). $\qquad\square$

Remark 4. Observe that Lemma 2.3 still works if we change ψ_j by $\psi_j\psi(|\xi_j|-2^k+1)$, $j = 1, 2$ or 3.

Proof of Proposition 2.2. Using the same notation as in Lemma 2.3, i.e., $\psi_j = \psi(2^{k+1} - |\xi_j|)$, $j = 1, 2, 3$, we define the sequence $\{\widetilde{\psi}_k\}$ as follows:

$$\widetilde{\psi}_0(\xi_1, \xi_2, \xi_3) = \psi(2 - |\xi_1|)\psi(2 - |\xi_2|)\psi(2 - |\xi_3|),$$

and for $k \geq 1$,

$$\bar{\psi}_k(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^{3} \psi_1\psi_2\psi_3\psi(|\xi_i| - 2^k + 1).$$

Notice that $\sum_{k\geq 0} \widetilde{\psi}_k = 1$.

Now we define the operator $\widehat{B_k f}(\xi) = \bar{\psi}_k^{1/2}(\xi)\hat{f}(\xi)$, $\xi \in \mathbb{R}^3$.

It is not difficult to verify that

$$\|B_k f\|_{L^2} \leq c2^{-ks}\|f\|_{H^s},$$
$$\widehat{B_k^2 f} = \bar{\psi}_k \hat{f},$$
$$\sum_{k\geq 0} \mathcal{E}(t)B_k^2 E_0 = \mathcal{E}(t)E_0,$$

and

$$|\int_{-T}^{T} (\mathcal{E}(t-\tau)(B_k^2 g(\cdot, \tau))(x, y, z)d\tau| \leq c(H_{k,T}(|\cdot|) * \int_{-T}^{T}\int\int|g(\tau, \cdot, y, z)|d\tau dy dz)(x),$$

for $|t| \leq T$ and $g \in C_0^\infty(\mathbb{R}^4)$.

Since from this point on the argument to complete the proof of the proposition is well understood (see for instance [8]) we will omit it. Thus the desired result follows. $\qquad\square$

Now, following ideas from [6] used to deal with solutions of the KPI equation, we show that (19) does not hold for $s < 1$.

Proposition 2.4. *For each $s < 1$ there exists F_0 such that*

$$\|\mathcal{E}(t)F_0\|_{L_x^2 L_{yzT}^\infty} \geq c(T, s)\|F_0\|_{H^s}.$$

Proof. Suppose that (19) is true and define $\hat{E}_0(\xi) = \hat{\theta}(\frac{\xi}{2^k})$, where $k \in \mathbb{N}$ and $\hat{\theta} \in C_0^\infty$ is such that

$$\hat{\theta}(\xi) = \begin{cases} 1 & \text{on} & \{\xi \in \mathbb{R}^3; 1 \leq |\xi| \leq 2\}, \\ 0 & \text{on} & \{\xi \in \mathbb{R}^3; |\xi| \leq 1/2\} \cup \{\xi \in \mathbb{R}^3; |\xi| \geq 4\}. \end{cases}$$

So by change of variables

$$\|E_0\|_{H^s}$$
$$= (\int_{\{\frac{1}{2} \leq \frac{|\xi|}{2^k} \leq 4\}} (1 + |\xi|^2)^s |\hat{\theta}(\frac{\xi}{2^k})|^2 d\xi)^{\frac{1}{2}} = (\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} (1 + 2^{2k}|\xi|^2)^s |\hat{\theta}(\xi)|^2 2^{3k} d\xi)^{\frac{1}{2}}$$
$$\leq 2^{3k/2+ks} (\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} (2^{-2k} + |\xi|^2)^s |\hat{\theta}(\xi)|^2 d\xi)^{\frac{1}{2}} \leq 2^{3k/2+ks} c(s) (\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} |\hat{\theta}(\xi)|^2 d\xi)^{\frac{1}{2}}$$
$$\leq 2^{3k/2+ks} c(s).$$

Next, we estimate $\mathcal{E}(t)E_0$. Again by changing variables we have

$$(\mathcal{E}(t)E_0)(x,y,z) = \int_{\{\frac{1}{2} \leq \frac{|\xi|}{2^k} \leq 4\}} e^{i(x\xi_1 + y\xi_2 + z\xi_3 + t(\xi_1^2 + \xi_2^2 + \xi_3))}\hat{\theta}(\frac{\xi}{2^k})d\xi$$

$$= 2^{3k} \int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} e^{ix\bar{\xi}}e^{is}\hat{\theta}(\xi)d\xi,$$

where $\bar{\xi} = 2^k\xi_1$, $s = y2^k\xi_2 + z2^k\xi_3 + t(2^{2k}\xi_1^2 + 2^{2k}\xi_2^2 + 2^k\xi_3)$.

Now, by Taylor's expansion

$$|(\mathcal{E}(t)E_0)(x,y,z)| = 2^{3k}|\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} e^{ix\bar{\xi}}e^{is}\hat{\theta}(\xi)d\xi|$$

$$\geq 2^{3k}|\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} (\cos(x\bar{\xi})\cos(s) - \sin(x\bar{\xi})\sin(s))\hat{\theta}(\xi)d\xi|$$

$$\geq 2^{3k}|\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} [(1 - \frac{(x\bar{\xi})^2}{2} + r(x\bar{\xi}))(1 - \frac{s^2}{2} + r(s)) +$$

$$- ((x\bar{\xi})^2 - r_1(x\bar{\xi}))(s - r_1(s))]\hat{\theta}(\xi)d\xi|$$

$$\geq 2^{3k}|\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} [1 - \eta(x,s,\bar{\xi}) + \rho(x,s,\bar{\xi})]\hat{\theta}(\xi)d\xi|,$$

where

$$\eta(x,s,\bar{\xi}) = \frac{(x\bar{\xi})^2}{2} + \frac{s^2}{2} + \frac{s^2 r(x\bar{\xi})}{2} + \frac{(x\bar{\xi})^2 r(s)}{2} + sx\bar{\xi} + r_1(x\bar{\xi})r_1(s),$$

$$\rho(x,s,\bar{\xi}) = r(x\bar{\xi}) + \frac{s^2(x\bar{\xi})^2}{2} + r(s) + r(x\bar{\xi})r(s) + x\bar{\xi}r_1(s) + sr_1(x\bar{\xi}),$$

$$r(\cdot) = (\cdot)^4 - (\cdot)^6 + (\cdot)^8 - \ldots \quad \text{and} \quad r_1(\cdot) = (\cdot)^3 - (\cdot)^5 + (\cdot)^7 - \ldots.$$

If we choose $0 < \delta \ll 1$ and take $|x| \leq \delta 2^{-k}$, y, $z \simeq \delta 2^{-k}$, $t \simeq \delta 2^{-2k}$, then s, $x\bar{\xi} \simeq O(\delta)$, $0 < r(s)$, $r_1(s)$, $r(x\bar{\xi})$, $r_1(x\bar{\xi}) \ll 1$ and $1 - \eta(x,s,\bar{\xi}) > c > 0$. So,

$$\|(\mathcal{E}(t)E_0)(x,y,z)| \geq c2^{3k}|\int_{\{\frac{1}{2} \leq |\xi| \leq 4\}} A \cdot \hat{\theta}(\xi)d\xi| \geq c2^{3k}|\int_{\{1 \leq |\xi| \leq 2\}} A \cdot 1 \, d\xi| \geq c2^{3k}.$$

Then,

$$\|(\mathcal{E}(t)E_0)\|_{L_x^2 L_{yzT}^\infty} \geq (\int_{|x| \leq \delta 2^{-k}} (\sup_{\substack{t \simeq \delta 2^{-2k} \\ y,z \simeq \delta 2^{-k}}} |E(t)E_0|)^2 dx)^{1/2} \geq 2^{3k}2^{-k/2} = 2^{5k/2}.$$

Finally, we have

$$c \, 2^{5k/2} \leq \|(\mathcal{E}(t)E_0)\|_{L_x^2 L_{yzT}^\infty} \leq \|E_0\|_{H^s} \leq 2^{3k/2 + ks} \; \forall k \in \mathbb{N},$$

which implies $s \geq 1$. □

Now we establish Strichartz estimates to the linear problem (14). Before that we state and prove an essential lemma:

Lemma 2.5. *If $t \neq 0, \frac{1}{p} + \frac{1}{p'} = 1$ and $p' \in [1,2]$, then the group $\mathcal{E}(t)$ defined in (15) is a continuous linear operator from $L_{xy}^{p'}L_z^2(\mathbb{R}^3)$ to $L_{xy}^p L_z^2(\mathbb{R}^3)$ and*

$$\|\mathcal{E}(t)f\|_{L_{xy}^p L_z^2} \leq \frac{c}{|t|^{(\frac{1}{p'} - \frac{1}{p})}}\|f\|_{L_{xy}^{p'}L_z^2}.$$

Proof. From Plancherel's theorem we have that

$$\|\mathcal{E}(t)f\|_{L^2_{xy}L^2_z} = \|\mathcal{E}(t)f\|_{L^2} = \|e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\hat{f}\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2_{xy}L^2_z}. \qquad (22)$$

Using Fourier's transform properties we obtain

$$(\mathcal{E}(t)f)(x,y,z)$$
$$= (e^{-it(\xi_1^2+\xi_2^2+\xi_3)}\widehat{f}(\xi_1,\xi_2,\xi_3))^{\vee}(x,y,z)$$
$$= \left(e^{-it\xi_3}(e^{-it(\xi_1^2+\xi_2^2)}\widehat{f}(\xi_1,\xi_2,\xi_3))^{\vee(x_1x_2)}(x,y,\cdot)\right)^{\vee(x_3)}(\cdot,\cdot,z)$$
$$= \left(e^{-it\xi_3}((e^{-it(\xi_1^2+\xi_2^2)})^{\vee(x_1x_2)} *_{x_1x_2} \widehat{f}^{(x_3)}(\xi_1,\xi_2,\xi_3))(x,y,\cdot)\right)^{\vee(x_3)}(\cdot,\cdot,z)$$
$$= \left(e^{-it\xi_3}(\frac{e^{i(\xi_1^2+\xi_2^2)/4|t|}}{4\pi t} *_{x_1x_2} \widehat{f}^{(x_3)}(\xi_1,\xi_2,\xi_3))(x,y,\cdot)\right)^{\vee(x_3)}(\cdot,\cdot,z)$$
$$= \left(e^{-it\xi_3}g(x,y,\xi_3)\right)^{\vee(x_3)}(\cdot,\cdot,z),$$

where

$$g(x,y,\cdot) = \left(\frac{e^{i(\xi_1^2+\xi_2^2)/4|t|}}{4\pi t} *_{x_1x_2} \widehat{f}^{(x_3)}(\xi_1,\xi_2,\xi_3)\right)(x,y,\cdot),$$

and $*_{x_1x_2}$ is the convolution in the first two variables, i.e.,

$$(f_1 *_{x_1x_2} f_2)(x,y,z) = \int_{\mathbb{R}^2} f_1(x-x_1,y-x_2,z)f_2(x_1,x_2,z)dx_1dx_2.$$

By Plancherel's theorem and Minkowski's inequality we have

$$\|\mathcal{E}(t)f(x,y,\cdot)\|_{L^2_z} = \|g(x,y,\cdot)\|_{L^2_z}$$
$$= \|\int\int \frac{e^{i((x-x_1)^2+(x-x_2)^2))/4|t|}}{4\pi t}\widehat{f}^{(x_3)}(x_1,x_2,\cdot)dx_1dx_2\|_{L^2_z}$$
$$\leq \int\int \|\frac{e^{i((x-x_1)^2+(x-x_2)^2))/4|t|}}{4\pi t}\widehat{f}^{(x_3)}(x_1,x_2,\cdot)\|_{L^2_z}dx_1dx_2$$
$$\leq \frac{1}{4\pi|t|}\int\int \|\widehat{f}^{(x_3)}(x_1,x_2,\cdot)\|_{L^2_z}dx_1dx_2$$
$$= \frac{1}{4\pi|t|}\int\int \|f(x_1,x_2,\cdot)\|_{L^2_z}dx_1dx_2 = \frac{1}{4\pi|t|}\|f\|_{L^1_{xy}L^2_z}.$$

Therefore, from the last inequality we obtain

$$\|\mathcal{E}(t)f\|_{L^\infty_{xy}L^2_z} \leq \frac{1}{4\pi|t|}\|f\|_{L^1_{xy}L^2_z}. \qquad (23)$$

Interpolation between inequalities (22) and (23) yields the result. □

Now we are able to prove Strichartz estimates. We notice that our result do not cover the endpoint $(p,q) = (\infty,2)$.

Proposition 2.6 (Strichartz estimates). *The unitary group $\{\mathcal{E}(t)\}_{-\infty}^{+\infty}$ defined in* (15) *satisfies*

$$\|\mathcal{E}(t)f\|_{L^q_t L^p_{xy}L^2_z} \leq c\|f\|_{L^2_{xyz}}, \qquad (24)$$

$$\|\int_{\mathbb{R}} \mathcal{E}(t-t')g(\cdot,t')dt'\|_{L^q_t L^p_{xy}L^2_z} \leq c\|g\|_{L^{q'}_t L^{p'}_{xy}L^2_z} \qquad (25)$$

and

$$\|\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,t)dt\|_{L^2_{xyz}} \leq c\|g\|_{L^{q'}_t L^{p'}_{xy}L^2_z}, \qquad (26)$$

where

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \ \frac{2}{q} = 1 - \frac{2}{p} \ and \ p = \frac{2}{\theta}, \ \theta \in (0, 1].$$

Proof. To prove this proposition we use standard by now arguments. First one shows that the three inequalities are equivalent. The main ingredient is the Stein-Thomas argument. Thus it is enough to establish for instance the estimate (25). To obtain (25) we use Lemma 2.5 and the Hardy-Littlewood-Sobolev theorem. $\square$

Next we recall some estimates proved in [10] regarding the solutions of the linear problem

$$\begin{cases} \partial_t^2 n + \Delta_\perp n = 0, & (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ n(\cdot, 0) = n_0(\cdot) \\ \partial_t n(\cdot, 0) = n_1(\cdot), \end{cases} \tag{27}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$. The solution of the problem (27) can be written as

$$n(\cdot, t) = N'(t)n_0 + N(t)n_1,$$

where $N(t)$ and $N'(t)$ were defined in (4) and (5).

Lemma 2.7. *For $f \in L^2(\mathbb{R}^3)$ we have*

$$\|N(t)f\|_{L^2(\mathbb{R}^3)} \le |t| \|f\|_{L^2(\mathbb{R}^3)}, \tag{28}$$

$$\|N'(t)f\|_{L^2(\mathbb{R}^3)} \le \|f\|_{L^2(\mathbb{R}^3)}, \tag{29}$$

and

$$\|(-\Delta_\perp)^{1/2} N(t)f\|_{L^2(\mathbb{R}^3)} \le \|f\|_{L^2(\mathbb{R}^3)}. \tag{30}$$

Remark 5. From this lemma one can easily deduce that

$$\sum_{|\alpha| \le 2} \|N(t)\partial^\alpha f\|_{L^2(\mathbb{R}^3)} \le c\|f\|_{H^1(\mathbb{R}^3)} + c|t|\|\partial_z f\|_{H^1(\mathbb{R}^3)}. \tag{31}$$

Lemma 2.8. *It holds that*

$$\|N'(t)n_0\|_{L_x^2 L_{yzT}^\infty} \le \|n_0\|_{H^2(\mathbb{R}^3)},$$

$$\|(-\Delta_\perp)^{1/2} N(t)n_1\|_{L_x^2 L_{yzT}^\infty} \le T\|n_1\|_{H^2(\mathbb{R}^3)},$$

and

$$\|N(t)n_1\|_{L_x^2 L_{yzT}^\infty} \le T(\|n_1\|_{H^1(\mathbb{R}^3)} + \|\partial_z n_1\|_{H^1(\mathbb{R}^3)}).$$

These estimates hold exchanging x and y.

In our argument we shall use some of the calculus inequalities involving fractional derivatives proved in [7]. More precisely, we recall the following estimate which is a particular case of those established in ([7], Theorem A.8).

Lemma 2.9. *Let $\rho \in (0, 1)$, $\rho_1, \rho_2 \in [0, \rho]$ with $\rho = \rho_1 + \rho_2$. Furthermore, let $p_1, p_2, q_1, q_2 \in [2, \infty)$ such that*

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|D_{x_j}^\rho (fg) - f D_{x_j}^\rho g - D_{x_j}^\rho f g\|_{L_{x_j}^2(\mathbb{R}; L^2(Q))} \le c \|D_{x_j}^{\rho_1} f\|_{L_{x_j}^{p_1}(\mathbb{R}; L^{q_1}(Q))} \|D_{x_j}^{\rho_2} g\|_{L_{x_j}^{p_2}(\mathbb{R}; L^{q_2}(Q))},$$

where $Q = \mathbb{R}^{n-1} \times [0, T]$.

3. **Nonlinear estimates.** In this section we will establish estimates for the non-linear terms involving in our analysis.

We begin by rewriting the integral equivalent form of the IVP (3) as

$$E(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')dt' + \int_0^t \mathcal{E}(t-t')(EL)(t')dt',$$

where

$$F(t) = N'(t)n_0 + N(t)n_1, \tag{32}$$

and

$$L(t) = \int_0^t N(t-t')\Delta_\perp(|E|^2)(t')dt'. \tag{33}$$

In the next lemma we treat the nonlinearity L in the Sobolev norm $\|\cdot\|_{H^2}$.

Lemma 3.1. *Let α, β_1, β_2 be multi-indices, then*

$$\sum_{|\alpha|\leq 2} \|\partial^\alpha L\|_{L^2_{xyz}} \leq c\,T^{1/2}\|E\|_{L^2_x L^\infty_{yzT}} \sum_{|\alpha|=2} \|\partial_x \partial^\alpha E\|_{L^\infty_x L^2_{yzT}}$$

$$+ c\,T^{1/2}\|E\|_{L^2_y L^\infty_{xzT}} \sum_{|\alpha|=2} \|\partial_y \partial^\alpha E\|_{L^\infty_y L^2_{xzT}}$$

$$+cT^{1/2}\sum_{\substack{|\beta_1|+|\beta_2|\leq 2\\ |\beta_2|=1}}\left(\|\partial_x\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z} + \|\partial_y\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z}\right)\|J_z^{\frac{1}{2}+}\partial^{\beta_2}\bar{E}\|_{L^4_{xyT}L^2_z} \tag{34}$$

$$+ cT\|E\|^2_{L^\infty_T H^2(\mathbb{R}^3)}.$$

Proof. Using the definition of L in (33) and the inequality (30) we have that

$$\sum_{|\alpha|\leq 2} \|\partial^\alpha L\|_{L^2_{xyz}} \leq c\sum_{|\alpha|\leq 2} \int_0^T \|(-\Delta_\perp)^{1/2}N(t'-s)(-\Delta_\perp)^{1/2}\partial^\alpha(|E|^2)(s)\|_{L^2_{xyzT}}ds$$

$$\leq c\sum_{|\alpha|\leq 2} \int_0^T \|(-\Delta_\perp)^{1/2}\partial^\alpha(|E|^2)(s)\|_{L^2_{xyz}}ds$$

$$\leq cT\|E\|^2_{L^\infty_T H^2(\mathbb{R}^3)} + cT^{1/2}\sum_{|\alpha|=2}\left(\|\partial_x\partial^\alpha(E\bar{E})\|_{L^2_{xyzT}} + \|\partial_y\partial^\alpha(E\bar{E})(s)\|_{L^2_{xyzT}}\right). \tag{35}$$

Next it will be enough to consider one of the terms inside the sum on the right hand side of (35). By Leibniz' rule we have

$$\|\partial_x\partial^\alpha(E\bar{E})\|_{L^2_{xyzT}} \leq c\sum_{|\beta_1|+|\beta_2|=2} \|\partial_x(\partial^{\beta_1}E\partial^{\beta_2}\bar{E})\|_{L^2_{xyzT}}$$

$$\leq c\sum_{|\beta_1|+|\beta_2|=2} \left(\|\partial_x\partial^{\beta_1}E\partial^{\beta_2}\bar{E}\|_{L^2_{xyzT}} + \|\partial^{\beta_1}E\partial_x\partial^{\beta_2}\bar{E}\|_{L^2_{xyzT}}\right). \tag{36}$$

Now we will consider just one of the terms on the right hand of the last inequality, the other one can be similarly treated. To simplify the exposition we choose the terms $\|\partial_x^3 E\bar{E}\|_{L^2_{xyzT}}$ and $\|\partial_x^2 E\partial_x\bar{E}\|_{L^2_{xyzT}}$ to show the next estimates since they have the same structure of the reminder terms in the sum in (36).

The Hölder inequality implies

$$\|\partial_x^3 E\bar{E}\|_{L^2_{xyzT}} \leq \|\partial_x^3 E\|_{L^\infty_x L^2_{yzT}}\|E\|_{L^2_x L^\infty_{xyzT}}. \tag{37}$$

On the other hand, using the Hölder inequality and the Sobolev lemma in the z-direction we obtain

$$
\begin{aligned}
\|\partial_x^2 E \partial_x \bar{E}\|_{L^2_{xyzT}} &\leq \|\|\partial_x^2 E\|_{L^2_z}\|\partial_x E\|_{L^\infty_z}\|_{L^2_{xyT}} \\
&\leq c \,\|\|\partial_x^2 E\|_{L^2_z}\|J_z^{\frac{1}{2}+}\partial_x E\|_{L^2_z}\|_{L^2_{xyT}} \\
&\leq c\|\partial_x^2 E\|_{L^4_{xyT}L^2_z}\|J_z^{\frac{1}{2}+}\partial_x E\|_{L^4_{xyT}L^2_z}.
\end{aligned}
\tag{38}
$$

Using the information in inequalities (37) and (38) in (36) and then in (35) the estimate (34) follows. □

Lemma 3.2. *Let α, β_1, β_2 be multiindexes, then*

$$
\begin{aligned}
\|L\|_{L^2_x L^\infty_{yzT}} &+ \|L\|_{L^2_y L^\infty_{xzT}} \leq cT^2\|E\|^2_{L^\infty_T H^2(\mathbb{R}^3)} \\
&+cT^{3/2}\|E\|_{L^2_x L^\infty_{yzT}}\sum_{|\alpha|=2}\|\partial_x\partial^\alpha E\|_{L^\infty_x L^2_{yzT}}+cT^{3/2}\|E\|_{L^2_y L^\infty_{xzT}}\sum_{|\alpha|=2}\|\partial_y\partial^\alpha E\|_{L^\infty_y L^2_{xzT}} \\
&+cT^{3/2}\sum_{\substack{|\beta_1|+|\beta_2|\leq 2 \\ |\beta_2|=1}}\Big(\|\partial_x\partial^{\beta_1} E\|_{L^4_{xyT}L^2_z}+\|\partial_y\partial^{\beta_1} E\|_{L^4_{xyT}L^2_z}\Big)\|J_z^{\frac{1}{2}+}\partial^{\beta_2}\bar{E}\|_{L^4_{xyT}L^2_z}.
\end{aligned}
$$

Proof. Using Lemma 2.7 we have that

$$
\begin{aligned}
\|L\|_{L^2_x L^\infty_{yzT}} &\leq \int_0^T \|(-\Delta_\perp)^{1/2}N(t-t')(-\Delta_\perp)^{1/2}(|E|^2)(t')\|_{L^2_x L^\infty_{yzT}}dt' \\
&\leq T\int_0^T \|(-\Delta_\perp)^{1/2}(|E|^2)(t')\|_{H^2}dt'.
\end{aligned}
$$

Thus applying the argument used in Lemma 3.1 the result follows. □

Lemma 3.3.

$$
\sum_{|\alpha|\leq 2}\|\partial^\alpha(EF)\|_{L^2_{xyzT}} \leq cT^{1/2}\|E\|_{L^\infty_T H^2}\big(\|n_0\|_{H^2}+\|n_1\|_{H^1}+T\|\partial_z n_1\|_{H^1}\big),
\tag{39}
$$

and

$$
\begin{aligned}
\sum_{|\alpha|\leq 2}\|\partial^\alpha(EL)\|_{L^2_{xyzT}} &\leq cT^{3/2}\|E\|^3_{L^\infty_T H^2(\mathbb{R}^3)} \\
&+cT^{1/2}\sum_{\substack{|\beta_1|+|\beta_2|\leq 2 \\ |\beta_2|=1}}\Big(\|\partial_x\partial^{\beta_1} E\|_{L^4_{xyT}L^2_z}+\|\partial_y\partial^{\beta_1} E\|_{L^4_{xyT}L^2_z}\Big)\|J_z^{\frac{1}{2}+}\partial^{\beta_2}\bar{E}\|_{L^4_{xyT}L^2_z} \\
&+cT\|E\|_{L^\infty_T H^2(\mathbb{R}^3)}\big(\|E\|_{L^2_x L^\infty_{yzT}}\sum_{|\alpha|=2}\|\partial_x\partial^\alpha E\|_{L^\infty_x L^2_{yzT}} \\
&+\|E\|_{L^2_y L^\infty_{xzT}}\sum_{|\alpha|=2}\|\partial_y\partial^\alpha E\|_{L^\infty_y L^2_{xzT}}\big).
\end{aligned}
\tag{40}
$$

Proof. To obtain the estimate (39) we first use properties of Sobolev spaces to obtain

$$
\sum_{|\alpha|\leq 2}\|\partial^\alpha(EF)\|_{L^2_{xyzT}} \leq cT^{1/2}\|E\|_{L^\infty_T H^2(\mathbb{R}^3)}\|F\|_{L^\infty_T H^2(\mathbb{R}^3)}.
$$

Then Lemma 2.7 and (31) yield the result.

Analogously to obtain the estimate (40) we first use properties of Sobolev spaces to obtain

$$\sum_{|\alpha|\leq 2}\|\partial^{\alpha}(EL)\|_{L^2_{xyzT}} \leq cT^{1/2}\|E\|_{L^\infty_T H^2(\mathbb{R}^3)}\|L\|_{L^\infty_T H^2(\mathbb{R}^3)}.$$

Then Lemma 3.1 yields the result.   □

Lemma 3.4. *It holds that*

$$\sum_{|\alpha|=2}\|\partial_x\int_0^t \mathcal{E}(t-t')\partial^{\alpha}(EF)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$\leq cT^{1/2}\|E\|_{L^\infty_T H^2}\big(\|n_0\|_{H^2}+T\|n_1\|_{H^1}+T\|\partial_z n_1\|_{H^1}\big)$$
$$+cT^{1/2}\|E\|_{L^2_x L^\infty_{yzT}}\big(\|n_0\|_{H^2}+\|n_1\|_{H^1}+T^{1/2}\|\partial_z n_1\|_{H^1}\big)$$
$$+\sum_{|\beta_1|=1}\big(T^{3/4}\|J_z^{1/4+}D_x^{1/2}\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z}+T^{5/8}\|J_z^{3/8+}\partial^{\beta_1}E\|_{L^{8/3}_T L^8_{xy}L^2_z}\big)$$
$$\times\big(\|n_0\|_{H^2}+T^{1/2}\|n_1\|_{H^1}+T\|\partial_z n_1\|_{H^1}\big).$$

These estimates holds exchanging x and y.

Proof. Let $\beta_i\in(\mathbb{Z}^+)^3$, $i=1,2$, be multi-indices. The Leibniz rule and Proposition 2.1 yield

$$\|\partial_x\int_0^t \mathcal{E}(t-t')\partial^{\alpha}(EF)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$\lesssim \sum_{|\alpha|=2}\|\partial_x\int_0^t \mathcal{E}(t-t')(\partial^{\alpha}E\,F+E\,\partial^{\alpha}F)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$+\sum_{|\beta_1|=|\beta_2|=1}\|\partial_x\int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L^\infty_x L^2_{yzT}} \tag{41}$$

$$\lesssim \sum_{|\alpha|=2}\big(\|\partial^{\alpha}E\,F\|_{L^1_x L^2_{yzT}}+\|E\,\partial^{\alpha}F\|_{L^1_x L^2_{yzT}}\big)$$

$$+\sum_{|\beta_1|=|\beta_2|=1}\int_0^T\|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}F)\|_{L^2_{xyz}}.$$

Using the Holder inequality Lemma 2.8 and Remark 5 we deduce that

$$\sum_{|\alpha|=2}\big(\|\partial^{\alpha}E\,F\|_{L^1_x L^2_{yzT}}+\|E\,\partial^{\alpha}F\|_{L^1_x L^2_{yzT}}\big)$$

$$\leq cT^{1/2}\|E\|_{L^\infty_T H^2}\big(\|n_0\|_{H^2}+T\|n_1\|_{H^1}+T\|\partial_z n_1\|_{H^1}\big) \tag{42}$$

$$+c\,\|E\|_{L^2_x L^\infty_{yzT}}\big(\|n_0\|_{H^2}+T^{1/2}\|n_1\|_{H^1(\mathbb{R}^3)}+T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big).$$

On the other hand, the use of the fractional Leibniz rule in Lemma 2.9, the Holder inequality and the Sobolev embedding in several stages yield the next chain

of inequalities

$$\sum_{|\beta_1|=|\beta_2|=1} \int_0^T \|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}F)\|_{L^2_{xyz}}$$

$$\leq c \sum_{|\beta_1|=|\beta_2|=1} \int_0^T \big(\|D_x^{1/2}\partial^{\beta_1}E(t')\|_{L^4_{xyz}}\|\partial^{\beta_2}F(t')\|_{L^4_{xyz}}$$

$$+ \|\partial^{\beta_1}E(t')D_x^{1/2}\partial^{\beta_2}F(t')\|_{L^2_{xyz}}\big)dt'$$

$$\leq c \sum_{|\beta_1|=1} \big(T^{3/4}\|J_z^{1/4+}D_x^{1/2}\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z}\|F\|_{L^\infty_T H^2} \tag{43}$$

$$+ T^{5/8}\|J_z^{3/8+}\partial^{\beta_1}E\|_{L^{8/3}_T L^8_{xy}L^2_z}\|F\|_{L^\infty_T H^2}\big).$$

$$\leq c \sum_{|\beta_1|=1} \big(T^{3/4}\|J_z^{1/4+}D_x^{1/2}\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z} + T^{5/8}\|J_z^{3/8+}\partial^{\beta_1}E\|_{L^{8/3}_T L^8_{xy}L^2_z}\big)$$

$$\times \big(\|n_0\|_{H^2} + T^{1/2}\|n_1\|_{H^1(\mathbb{R}^3)} + T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big).$$

Thus combining (42), (43) and (41) the result follows. □

Lemma 3.5. *It holds that*

$$\sum_{|\alpha|=2} \|\partial_x \int_0^t \mathcal{E}(t-t')\partial^\alpha(EL)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$\leq cT^{1/2}\|E\|_{L^\infty_T H^2}\|L\|_{L^2_x L^\infty_{yzT}} + cT^{1/2}\|E\|_{L^2_x L^\infty_{yzT}}\|L\|_{L^\infty_T H^2}.$$

$$+ c\sum_{|\beta_1|=1} \big(T^{3/4}\|J_z^{1/4+}D_x^{1/2}\partial^{\beta_1}E\|_{L^4_{xyT}L^2_z} + T^{5/8}\|J_z^{3/8+}\partial^{\beta_1}E\|_{L^{8/3}_T L^8_{xy}L^2_z}\big)\|L\|_{L^\infty_T H^2}.$$

The estimate holds exchanging x and y.

Proof. We follow the argument in the previous lemma. More precisely, Let $\beta_i \in (\mathbb{Z}^+)^3$, $i = 1, 2$, be multi-indices. The Leibniz rule and Proposition 2.1 yield

$$\|\partial_x \int_0^t \mathcal{E}(t-t')\partial^\alpha(EL)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$\lesssim \sum_{|\alpha|=2} \|\partial_x \int_0^t \mathcal{E}(t-t')(\partial^\alpha E\, L + E\, \partial^\alpha L)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$+ \sum_{|\beta_1|=|\beta_2|=1} \|\partial_x \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}L)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$\lesssim \sum_{|\alpha|=2} \big(\|\partial^\alpha E\, L\|_{L^1_x L^2_{yzT}} + \|E\, \partial^\alpha L\|_{L^1_x L^2_{yzT}}\big)$$

$$+ \sum_{|\beta_1|=|\beta_2|=1} \int_0^T \|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}L)\|_{L^2_{xyz}}$$

Using the Holder inequality we deduce that

$$\sum_{|\alpha|=2} \big(\|\partial^\alpha E\, L\|_{L^1_x L^2_{yzT}} + \|E\, \partial^\alpha L\|_{L^1_x L^2_{yzT}}\big)$$

$$\leq cT^{1/2}\|E\|_{L^\infty_T H^2}\|L\|_{L^2_x L^\infty_{yzT}} + cT^{1/2}\|E\|_{L^2_x L^\infty_{yzT}}\|L\|_{L^\infty_T H^2}.$$

On the other hand, the use of the fractional Leibniz rule in Lemma 2.9, the Holder inequality and the Sobolev embedding yield

$$
\sum_{|\beta_1|=|\beta_2|=1} \int_0^T \|D_x^{1/2}(\partial^{\beta_1} E \partial^{\beta_2} L)\|_{L_{xyz}^2}
$$

$$
\leq c \sum_{|\beta_1|=|\beta_2|=1} \int_0^T \left(\|D_x^{1/2}\partial^{\beta_1} E(t')\|_{L_{xyz}^4} \|\partial^{\beta_2} L(t')\|_{L_{xyz}^4} \right.
$$

$$
+ \|\partial^{\beta_1} E(t') D_x^{1/2}\partial^{\beta_2} L(t')\|_{L_{xyz}^2} \Big) dt'
$$

$$
\leq c \sum_{|\beta_1|=1} \left(T^{3/4}\|J_z^{1/4+}D_x^{1/2}\partial^{\beta_1} E\|_{L_{xyT}^4 L_z^2} + T^{5/8}\|J_z^{3/8+}\partial^{\beta_1} E\|_{L_T^{8/3} L_{xy}^8 L_z^2} \right) \|L\|_{L_T^\infty H^2}.
$$

$\square$

4. **Proof of Theorem 1.1.** As we mention in the introduction we will use the contraction mapping principle.

We first define the metric space

$$
X_{a,T} = \{ E \in C([0,T] : \widetilde{H}^2(\mathbb{R}^3)) : \|E\| \leq a \},
$$

where

$$
\|E\| := \|E\|_{L_T^\infty H^2(\mathbb{R}^3)} + \sum_{|\alpha|=2} \left(\|D_x^{1/2}\partial^\alpha E\|_{L_T^\infty L_{xyz}^2} + \|D_y^{1/2}\partial^\alpha E\|_{L_T^\infty L_{xyz}^2} \right)
$$

$$
+ \sum_{|\alpha|=1} \left(\|J_z^{1/4+}D_x^{1/2}\partial^\alpha E\|_{L_{xyT}^4 L_z^2} + \|J_z^{3/8+}\partial^\alpha E\|_{L_T^{8/3} L_{xy}^8 L_z^2} \right.
$$

$$
+ \|J_z^{1/2+}\partial^\alpha E\|_{L_{xyT}^4 L_z^2} \Big) + \|E\|_{L_x^2 L_{yzT}^\infty} + \|E\|_{L_y^2 L_{xzT}^\infty}
$$

$$
+ \sum_{|\alpha|\leq 1} \left(\|\partial_x \partial^\alpha E\|_{L_{xyT}^4 L_z^2} + \|\partial_y \partial^\alpha E\|_{L_{xyT}^4 L_z^2} \right)
$$

$$
+ \sum_{|\alpha|=2} \left(\|\partial_x \partial^\alpha E\|_{L_x^\infty L_{yzT}^2} + \|\partial_y \partial^\alpha E\|_{L_y^\infty L_{xzT}^2} \right).
$$

and the integral operator on $X_{a,T}$,

$$
\Psi(E)(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')dt' + \int_0^t \mathcal{E}(t-t')(EL)(t')dt', \quad (44)
$$

where F and L were defined in (32) and (33), respectively.

We will show that for appropriate a and T the operator $\Psi(\cdot)$ defines a contraction on $X_{a,T}$.

We begin by estimating the $H^2(\mathbb{R}^3)$-norm of $\Psi(E)$. Let $E \in X_{a,T}$. By Fubini's Theorem, Minkowski's inequality and group properties we have

$$
\|\Psi(E)(t)\|_{H^2} \leq \|E_0\|_{H^2} + \|E\|_{L_T^\infty H^2} \int_0^T \|F(t')\|_{H^2} dt'
$$

$$
+ \|E\|_{L_T^\infty H^2} \int_0^T \|L(t')\|_{H^2} dt'. \tag{45}
$$

From Lemma 2.7, Lemma 3.1 and inequality (31) we have

$$
\begin{aligned}
\|\Psi(E)(t)\|_{H^2} \leq\ &\|E_0\|_{H^2} + cT\|E\|_{L_T^\infty H^2}\big(\|n_0\|_{H^2} + \|n_1\|_{H^1(\mathbb{R}^3)} + T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big) \\
&+ cT^{3/2}\|E\|_{L_T^\infty H^2(\mathbb{R}^3)}^3 + cT^{3/2}\|E\|_{L_T^\infty H^2}\|E\|_{L_x^2 L_{yzT}^\infty} \sum_{|\alpha|=2} \|\partial_x \partial^\alpha E\|_{L_x^\infty L_{yzT}^2} \\
&+ c\,T^{3/2}\|E\|_{L_T^\infty H^2}\|E\|_{L_y^2 L_{xzT}^\infty} \sum_{|\alpha|=2} \|\partial_y \partial^\alpha E\|_{L_y^\infty L_{xzT}^2} \\
&+ cT^{3/2}\|E\|_{L_T^\infty H^2} \times \\
&\sum_{\substack{|\beta_1|+|\beta_2|\leq 2 \\ |\beta_2|=1}} \Big(\|\partial_x \partial^{\beta_1} E\|_{L_{xyT}^4 L_z^2} + \|\partial_y \partial^{\beta_1} E\|_{L_{xyT}^4 L_z^2}\Big)\|J_z^{\frac{1}{2}+}\partial^{\beta_2}\bar{E}\|_{L_{xyT}^4 L_z^2}.
\end{aligned}
$$

(46)

Therefore

$$
\begin{aligned}
\|\Psi(E)(t)\|_{L_T^\infty H^2} \leq\ &\|E_0\|_{H^2} + cT\|E\|\big(\|n_0\|_{H^2} + \|n_1\|_{H^1(\mathbb{R}^3)} + T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big) \\
&+ cT^{3/2}\|E\|^3.
\end{aligned}
$$

Next, we estimate the norms

$$
\|\cdot\|_{L_x^2 L_{yzT}^\infty}, \quad \sum_{|\alpha|\leq 2} \|\partial_x \partial^\alpha \cdot\|_{L_x^\infty L_{yzT}^2}, \quad \|\cdot\|_{L_y^2 L_{xzT}^\infty}, \quad \sum_{|\alpha|\leq 2} \|\partial_y \partial^\alpha \cdot\|_{L_y^\infty L_{xzT}^2}.
$$

By symmetry is enough to estimate the first two norms. Thus, using the definition of Ψ in (44), Proposition 2.2 and the inequalities (45) and (46) it follows that

$$
\begin{aligned}
\|\Psi(E)\|_{L_x^2 L_{yzT}^\infty} \leq\ &\|E_0\|_{H^2} + cT\|E\|\big(\|n_0\|_{H^2} + \|n_1\|_{H^1(\mathbb{R}^3)} + T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big) \\
&+ cT^{3/2}\|E\|^3.
\end{aligned}
$$

(47)

Next we use Proposition 2.6 and then the inequalities (45) and (46) to obtain

$$
\begin{aligned}
&\sum_{|\alpha|=1} \|J_z^{1/4+} D_x^{1/2}\partial^\alpha \Psi(E)\|_{L_{xyT}^4 L_z^2} + \|J_z^{3/8+}\partial^\alpha \Psi(E)\|_{L_T^{8/3} L_{xy}^8 L_z^2} \\
&+ \|J_z^{1/2+}\partial^\alpha \Psi(E)\|_{L_{xyT}^4 L_z^2} + \sum_{|\alpha|\leq 1} \big(\|\partial_x \partial^\alpha \Psi(E)\|_{L_{xyT}^4 L_z^2} + \|\partial_y \partial^\alpha \Psi(E)\|_{L_{xyT}^4 L_z^2}\big) \\
&\leq c\big(\|E_0\|_{H^2} + \|E\|_{L_T^\infty H^2}\int_0^T \|F(t')\|_{H^2}dt' + \|E\|_{L_T^\infty H^2}\int_0^T \|L(t')\|_{H^2}dt'\big). \\
&\leq \|E_0\|_{H^2} + cT\|E\|\big(\|n_0\|_{H^2} + \|n_1\|_{H^1(\mathbb{R}^3)} + T\|\partial_z n_1\|_{H^1(\mathbb{R}^3)}\big) + cT^{3/2}\|E\|^3.
\end{aligned}
$$

Now using the definition of Ψ in (44), Proposition 2.1 and Lemmas 3.4 and 3.5 we obtain

$$
\begin{aligned}
\sum_{|\alpha|=2} \|\partial_x \partial^\alpha \Psi(E)\|_{L_x^\infty L_{yzT}^2} \leq\ &c \sum_{|\alpha|=2} \|D_x^{1/2}\partial^\alpha E_0\|_{L^2} + c(T)\,T\,\|E\|^3 \\
&+ c(T)\,T^{1/2}\,\|E\|\big(\|n_0\|_{H^2} + \|n_1\|_{H^1} + T\|\partial_z n_1\|_{H^1}\big)
\end{aligned}
$$

It remains only to estimate the norms

$$
\sum_{|\alpha|=2} \|D_x^{1/2}\partial^\alpha \cdot\|_{L_T^\infty L_{xyz}^2} \quad \text{and} \quad \sum_{|\alpha|=2} \|D_y^{1/2}\partial^\alpha \cdot\|_{L_T^\infty L_{xyz}^2}.
$$

Once again by symmetry we only estimate the first one.

From the definition of Ψ in (44), Proposition 2.1 and Lemmas 3.4 and 3.5 we get

$$\sum_{|\alpha|=2} \|D_x^{1/2}\partial^\alpha \Psi(E)\|_{L_T^\infty L_{xyz}^2} \leq \sum_{|\alpha|=2} \|D_x^{1/2}\partial^\alpha E_0\|_{L^2} + c(T)\,T\,\|\!|E|\!\|^3$$
$$+ c(T)\,T^{1/2}\,\|\!|E|\!\| \left(\|n_0\|_{H^2} + \|n_1\|_{H^1} + T\|\partial_z n_1\|_{H^1}\right).$$

Hence, a suitable choice of $a = a(\|E_0\|_{\tilde{H}^3}, T)$ and T (T sufficiently small depending on $\|n_0\|_{\widetilde{H}^3}$, $\|n_1\|_{\widetilde{H}^2}$ and $\|\partial_z n_1\|_{\widetilde{H}^2}$), we see that Ψ maps $X_{a,T}$ into $X_{a,T}$.

Since the reminder of the proof follows a similar argument we will omit it.

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