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THE DAVEY STEWARTSON SYSTEM IN WEAK L^p SPACES

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Abstract. We study the global Cauchy problem associated to the Davey-Stewartson system in $\mathbb{R}^n,\ n = 2, 3$. Existence and uniqueness of the solution are established for small data in some weak L^p space. We apply an interpolation theorem and the generalization of the Strichartz estimates for the Schrödinger equation derived in [9]. As a consequence we obtain self-similar solutions.

1. Introduction

This paper is concerned with the initial value problem (IVP) associated to the Davey-Stewartson system

$$\begin{cases} i\partial_t u + \delta\partial^2_{x_1} u + \sum_{j=2}^{n} \partial^2_{x_j} u = \chi|u|^\alpha u + bu\partial_{x_1}\varphi, \\ \partial^2_{x_1}\varphi + m\partial^2_{x_2}\varphi + \sum_{j=3}^{n} \partial^2_{x_j}\varphi = \partial_{x_1}(|u|^\alpha), \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

$(x,t) \in \mathbb{R}^n \times \mathbb{R},\ n = 2, 3$, where $u = u(x,t)$ is a complex-valued function and $\varphi = \varphi(x,t)$ is a real-valued function.

The exponent α is such that $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$, the parameters χ and b are constants in $\mathbb{R}$, δ and m are real positive, and we can consider δ and χ normalized in such a way that $|\delta| = |\chi| = 1$.

The Davey-Stewartson systems are $2D$ generalizations of the cubic $1D$ Schrödinger equation,

$$i\partial_t u + \Delta u = |u|^2 u$$

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and model the evolution of weakly nonlinear water waves that travel predominantly in one direction but for which the amplitude is modulated slowly in two horizontal directions.

System (1.1), $n = 2,\ \alpha = 2$, was first derived by Davey and Stewartson [13] in the context of water waves, but its analysis did not take into account the effect of surface tension (or capillarity). This effect was later included by Djordjevic and Redekopp [12], who have shown that the parameter m can become negative when capillary effects are important. Independently, Ablowitz and Haberman [3] obtained a particular form of (1.1), $n = 2$, as an example of a completely integrable model also generalizing the two-dimensional nonlinear Schrödinger equation.

When $(\delta, \chi, b, m) = (1, -1, 2, -1), (-1 - 2, 1, 1), (-1, 2, -1, 1)$, the system (1.1), $n = 2$, is referred to as DSI, $DSII$ defocusing, and $DSII$ focusing respectively in the inverse scattering literature. In these cases several results concerning the existence of solutions or lump solutions have been established ([1], [2], [4], [6], [15], [16], [36]) by the inverse scattering techniques.

In [18], Ghidaglia and Saut studied the existence of solutions of IVP (1.1), $n = 2,\ \alpha = 2$. They classified the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic, according to respective sign of (δ, m): $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$.

For the elliptic-elliptic and hyperbolic-elliptic cases, Ghidaglia and Saut [18] reduced the system (1.1), $n = 2$, to the nonlinear cubic Schrödinger equation with a nonlocal nonlinear term, i.e.,

$$i\partial_t u + \delta \partial_{x_1}^2 u + \partial_{x_2}^2 u = \chi |u|^2 u + H(u),$$

where $H(u) = (\Delta^{-1}\partial_x^2 |u|^2)u$. They showed local well-posedness for data in L^2, H^1, and H^2 using Strichartz estimates and the continuity properties of the operator Δ^{-1}.

The remaining cases, elliptic-hyperbolic and hyperbolic-hyperbolic, were treated by Linares and Ponce [28]; Hayashi [19], [20]; Chihara [7]; Hayashi and Hirata [21], [22]; and Hayashi and Saut [23] (see [29] for further references).

Here we will concentrate on the elliptic-elliptic and hyperbolic-elliptic cases. We start with the motivation for this work:

From the condition $m > 0$ we are allowed to reduce the Davey-Stewartson system (1.1) to the Schrödinger equation

$$i\partial_t u + \delta \partial_{x_1}^2 u + \sum_{j=2}^{n} \partial_{x_j}^2 u = \chi |u|^\alpha u + buE(|u|^\alpha), \qquad u(x,0) = u_0(x), \tag{1.2}$$

$\forall\, x \in \mathbb{R}^n,\ n = 2, 3,\ t \in \mathbb{R}$, where

$$\widehat{E(f)}(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^n \xi_j^2} \hat{f}(\xi) = p(\xi)\hat{f}(\xi). \tag{1.3}$$

Now observe that if $u(x,t)$ satisfies

$$iu_t + \delta \partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi |u|^\alpha u + buE(|u|^\alpha),$$

then $u_\beta(x,t) = \beta^{2/\alpha} u(\beta x, \beta^2 t)$ does also, for all $\beta > 0$.

Therefore, it is natural to ask whether solutions $u(x,t)$ of (1.1) exist and satisfy, for $\beta > 0$,

$$u(x,t) = \beta^{2/\alpha} u(\beta x, \beta^2 t).$$

Such solutions are called self-similar solutions of the equation (1.2).

Therefore, supposing local well posedness and u a self-similar solution we must have

$$u(x,0) = u_\beta(x,0),\ \forall\ \beta > 0; \quad \text{i.e.,} \quad u_0(x) = \beta^{2/\alpha} u_0(\beta x).$$

In other words, $u_0(x)$ is homogeneous with degree $-2/\alpha$ and every initial data that gives a self-similar solution must satisfy this property. Unfortunately, those functions do not belong to the usual spaces where strong solutions exist, such as the Sobolev spaces $H^s(\mathbb{R}^n)$. We shall therefore replace them by other functional spaces that allow homogeneous functions.

There are many motivations to find self-similar solutions. One of them is that they can give a good description of the large-time behaviour for solutions of dispersive equations.

The idea of constructing self-similar solutions by solving the initial value problem for homogeneous data was first used by Giga and Miyakawa [17], for the Navier-Stokes equation in vorticity form. The idea of [17] was used later by Cannone and Planchon [8] and Planchon [32] (for the Navier-Stokes equation); Kwak [24] and Snoussi, Tayachi, and Weissler [37] (for nonlinear parabolic problems); Kavian and Weissler [25], Pecher [33], and Ribaud and Youssfi [35] (for the nonlinear wave equation); and Cazenave and Weissler [10], [11], Ribaud and Youssfi [34], and Furioli [14] (for the nonlinear Schrödinger equation).

In [9] Cazenave, Vega, and Vilela studied the global Cauchy problem for the Schrödinger equation

$$i\partial_t u + \Delta u = \gamma |u|^\alpha u, \qquad \alpha > 0,\ \gamma \in \mathbb{R},\ (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.4}$$

Using a generalization of the Strichartz estimates for the Schrödinger equation they showed that, under some restrictions on α, if the initial value is sufficiently small in some weak L^p space then there exists a global solution. This result provided a common framework to the classical H^s solutions and to the self-similar solutions. We follow their ideas in our work. From the condition $m > 0$ we are allowed to reduce the Davey-Stewartson system (1.1) to the Schrödinger equation (1.2). Now comparing Schrödinger equations (1.2) and (1.4) we observe that we have the nonlocal term $uE(|u|^2)$ to treat. The main ingredient to do that will be an interpolation theorem and the generalization of the Strichartz estimates for the Schrödinger equation derived in [9]. As a consequence, we prove existence and uniqueness (in the sense of distributions) to the IVP problem (1.2). As a consequence we find self-similar solutions for the problem (1.2) in the case $\delta > 0$.

To study the IVP (1.2) we use its integral equivalent formulation

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)(\chi|u|^\alpha u + buE(|u|^\alpha))(s)ds, \tag{1.5}$$

where $U(t)u_0$, defined as

$$\widehat{U(t)u_0}(\xi) = e^{-it\psi(\xi)}\widehat{u_0}(\xi), \qquad \psi(\xi) = 4\pi^2\delta\xi_1^2 + 4\pi^2\sum_{j=2}^n \xi_j^2, \tag{1.6}$$

is the solution of the linear problem associated to (1.2).

We also define the subspace $Y \subset S'(\mathbb{R}^n)$, where

$$Y = \{\varphi \in S'(\mathbb{R}^n) : U(t)\varphi \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})\},$$

$$\|\varphi\|_Y = \|U(t)\varphi\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}, \quad \text{and} \quad L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$$

are weak L^p spaces that we define later.

Our main result in this paper reads as follows:

Theorem 1. *There exists $\delta_1 > 0$ such that given*

$$\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$$

and $u_0 \in Y$ with $\|u_0\|_Y < \delta_1$, then there exists a unique solution $u \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$ of (1.5) such that $\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < 3\delta_1$.

To obtain this result we will use the contraction-mapping theorem and some estimates for the nonlocal operator E, defined in (1.3).

As a consequence of Theorem 1, we show that given any initial data in Y and assuming the existence of a solution u to the integral equation (1.5) we have that u is the solution (in the weak sense) of the differential equation (1.2). We emphasize that Theorem 1 provides the existence of solutions to the equation (1.5) under the assumption of small initial data.

Proposition 2. *Suppose*

$$\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2},$$

$u_0 \in Y$, *and let* $u \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$ *be the solution of* (1.5)*. It follows that* $t \in \mathbb{R} \to u(t) \in S'(\mathbb{R}^n)$ *is continuous and* $u(0) = u_0$*. In particular, u is a solution of* (1.2)*. Moreover,* $u(t_0) \in Y$ *for all* $t_0 \in \mathbb{R}$*. In addition, there exist* $u_\pm$ *such that* $\|U(t)u_\pm\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \infty$ *and* $U(-t)u(t) \to u_\pm$ *in* $S'(\mathbb{R}^n)$ *as* $t \to \pm\infty$.

Remark 3. We notice that Theorem 1 works for the hyperbolic-elliptic case. As far as we know, the existence of self-similar solutions in this case is an open problem. Recently, Kevrekidis, Nahmod, and Zeng [26] introduced a method to prove the existence of self-similar solutions for the hyperbolic cubic Schrödinger equation. It would be interesting to investigate whether this method can be applied to obtain self-similar solutions for the hyperbolic-elliptic case.

Remark 4. We also observe that the analysis developed here will work for the Zakharov-Schulman system, i.e.,

$$\begin{cases} i\partial_t u + L_1 u = \varphi u, \\ L_2\varphi = L_3(|u|^\alpha), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1.7}$$

where $u = u(x,t)$ is a complex-valued function, $\varphi = \varphi(x,t)$ is a real-valued function, and

$$L_j = \sum_{i=1}^{n} a_j^i \partial^2_{x_i x_i}, \quad k = 1,2,3,$$

when the operator L_2 is elliptic; see [27] and [31].

This paper is organized as follows: In Section 2 we show the main theorem. In preparation for that we will establish some needed estimates for the integral operator. The last section will be devoted to finding self-similar solutions.

2. Global existence in weak L^p spaces

Let us define the weak L^p spaces we will use in the following:

Definition 5.

$$L^{p\infty}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \text{ measurable } \|f\|_{L^{p\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda\alpha(\lambda, f)^{1/p} < \infty\},$$

where $\alpha(\lambda, f) = \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$ *and* $\mu =$ *Lebesgue measure.*

The reader should refer to [5] for details.

Remark 6. Using a change of variables it is easy to see that for any $\varphi \in S'(\mathbb{R}^n)$, $1 \le p \le \infty$, and $\tau \in \mathbb{R}$,

$$\|U(t)\varphi\|_{L^{p\infty}(\mathbb{R}^{n+1})} = \|U(t+\tau)\varphi\|_{L^{p\infty}(\mathbb{R}^{n+1})},$$

where $U(t)$ is the unitary group defined in (1.6).

The next theorem establishes a relationship between Lorentz spaces $L^{p\infty}$ and L^q spaces:

Theorem 7 (Interpolation theorem)**.** *Given* $0 < p_0 < p_1 \le \infty$*, for all* p *and* θ *such that* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $0 < \theta < 1$*, we have*

$$(L^{p_0}, L^{p_1})_{\theta,\infty} = L^{p\infty} \quad \text{with} \quad \|f\|_{(L^{p_0},L^{p_1})_{\theta,\infty}} = \|f\|_{L^{p\infty}},$$

where

$$(L^{p_0}, L^{p_1})_{\theta,\infty} = \Big\{\text{Lebesgue-measurable } a : \|a\|_{(L^{p_0},L^{p_1})_{\theta,\infty}} := \sup_{t>0} t^{-\theta} k(t,a) < \infty\Big\}$$

and

$$k(t,a) = \inf_{a=a_0+a_1} (\|a_0\|_{L^{p_0}} + t\|a_1\|_{L^{p_1}}).$$

Proof. We refer to [5] for a proof of this theorem. □

Remark 8. Another relationship between Lorentz spaces and L^p spaces is given by the following decomposition: Let $1 \le p_1 < p < p_2 < \infty$. Then

$$L^{p\infty} = L^{p_1} + L^{p_2}.$$

The next result is a generalization of the classical Strichartz estimates for the Schrödinger equation. This was proved by Vilela in [30].

Theorem 9. *Consider r, $\tilde{r}$, q, and $\tilde{q}$ such that*

$$2 < r, \tilde{r} \leq \infty, \quad \frac{1}{\tilde{r}'} - \frac{1}{r} < \frac{2}{n}, \qquad \frac{1}{\tilde{q}'} - \frac{1}{q} + \frac{n}{2}\Big(\frac{1}{\tilde{r}'} - \frac{1}{r}\Big) = 1, \tag{2.1}$$

$$\begin{cases} r, \tilde{r} \neq \infty & \text{if } n = 2, \\ \frac{n-2}{n}\Big(1 - \frac{1}{\tilde{r}'}\Big) \leq \frac{1}{r} \leq \frac{n}{n-2}\Big(1 - \frac{1}{\tilde{r}'}\Big) & \text{if } n \geq 3, \end{cases} \tag{2.2}$$

and

$$\begin{cases} 0 < \frac{1}{q} \leq \frac{1}{\tilde{q}'} < 1 - \frac{n}{2}\Big(\frac{1}{\tilde{r}'} + \frac{1}{r} - 1\Big) & \text{if } \frac{1}{\tilde{r}'} + \frac{1}{r} \geq 1, \\ -\frac{n}{2}\Big(\frac{1}{\tilde{r}'} + \frac{1}{r} - 1\Big) < \frac{1}{q} \leq \frac{1}{\tilde{q}'} < 1 & \text{if } \frac{1}{\tilde{r}'} + \frac{1}{r} < 1. \end{cases}$$

Then we have the following inequalities:

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \Big\|_{L_t^q L_x^r} \leq c \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{2.3}$$

$$\Big\| \int_{-\infty}^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \Big\|_{L_t^q L_x^r} \leq c \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

$$\Big\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \Big\|_{L_t^q L_x^r} \leq c \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Proof. We refer to [30] for a proof of this theorem. □

Remark 10. Theorem 9 also holds for $U(t)$.

To prove Theorem 1 we need some results:

Proposition 11. *Consider $F : \mathbb{R}_x^n \times \mathbb{R}_t \to \mathbb{C}$. Then for $1 < p < \infty$,*

$$\|E(F)\|_{L^{p\infty}(\mathbb{R}^{n+1})} \leq \|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}.$$

Instead of proving Proposition 11 we establish a more general result:

Lemma 12. *Let A be a linear injective operator and suppose that for each $1 \leq p < \infty$ there exists $1 \leq q = q(p) < \infty$ such that $A : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded. Then A is bounded from $L^{p\infty}(\mathbb{R}^n)$ to $L^{q\infty}(\mathbb{R}^n)$.*

Proof. In fact, fix $1 \leq p < \infty$. Take $1 \leq p_0$, $p_1 < \infty$, and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By Theorem 7 we have $\|A(f)\|_{L^{p\infty}(\mathbb{R}^n)} = \|A(f)\|_{(L^{p_0}, L^{p_1})_{\theta\infty}}$.

If $f = f_0 + f_1 \in L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$, then

$$A(f) = A(f_0) + A(f_1) \in L^{q_0}(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n),$$

and

$$\|A(f_j)\|_{L^{q_j}(\mathbb{R}^n)} \leq \|f_j\|_{L^{p_j}(\mathbb{R}^n)},\ j=0,1.$$

So

$$\begin{aligned} K(t,A(f)) &= \inf_{A(f)=F_0+F_1}(\|F_0\|_{L^{q_0}(\mathbb{R}^n)}+t\|F_1\|_{L^{q_1}(\mathbb{R}^n)}) \\ &\leq \inf_{A(f)=A(f_0)+A(f_1)}(\|A(f_0)\|_{L^{q_0}(\mathbb{R}^n)}+t\|A(f_1)\|_{L^{q_1}(\mathbb{R}^n)}) \\ &\leq \inf_{A(f)=A(f_0)+A(f_1)}(\|f_0\|_{L^{p_0}(\mathbb{R}^n)}+t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}). \end{aligned}$$

Since A is injective, $A(f)=A(f_0)+A(f_1)\Rightarrow f=f_0+f_1$ Lebesgue almost everywhere. Then

$$K(t,A(f)) \leq \inf_{f=f_0+f_1}(\|f_0\|_{L^{p_0}(\mathbb{R}^n)}+t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}) = K(t,f).$$

Using Theorem 7 once more we obtain the result. □

Observe that since the linear operator E defined in (1.3) is injective and satisfies

$$\|E(F)\|_{L^q(\mathbb{R}^{n+1})} \leq \|F\|_{L^q(\mathbb{R}^{n+1})}$$

for all $1<q<\infty$ (see [38]), Proposition 11 will be a consequence of Lemma 12. Now we define two integral operators:

$$G(F)(x,t) = \int_0^t U(t-s)F(\cdot,s)(x)ds, \tag{2.4}$$

and

$$(TT^*F)(x,t) = \int_{-\infty}^{+\infty} U(t-\tau)F(x,\tau)d\tau, \tag{2.5}$$

where $U(t)$ is the group defined in (1.6). We prove the following properties about them:

Proposition 13. *Let $1\leq p,r<\infty$ such that*

$$\frac{1}{p}-\frac{1}{r}=\frac{2}{n+2},$$

and

$$\frac{2(n+1)}{n} < r < \frac{2(n+1)(n+2)}{n^2}.$$

Then

$$\|G(F)\|_{L^{r\infty}(\mathbb{R}^{n+1})} \leq c\|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}, \tag{2.6}$$

and

$$\|TT^*(F)\|_{L^{r\infty}(\mathbb{R}^{n+1})} \leq c\|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}. \tag{2.7}$$

Proof. To prove Property (2.6) we need Theorem 9 (with $U(t)$ instead of $e^{it\Delta}$) and the interpolation theorem. In fact, taking $r = q$ and $\tilde{r}' = \tilde{q}' =: p$ in Theorem 9, the hypothesis (2.1) becomes

$$\frac{1}{p} - \frac{1}{r} = \frac{2}{n+2},$$

and the inequality (2.3) becomes

$$\|G(F)\|_{L^r(\mathbb{R}^{n+1})} \leq c\|F\|_{L^p(\mathbb{R}^{n+1})}. \tag{2.8}$$

The restriction $\frac{2(n+1)}{n} < r < \frac{2(n+1)(n+2)}{n^2}$ comes from hypothesis (2.2).

The result follows applying Lemma 12 to inequality (2.8). Property (2.7) is proved exactly the same way. □

Now we are ready to prove the our main result:

Proof of Theorem 1. Consider the following operator:

$$(\Phi u)(t) = U(t)u_0 - iG(\chi|u|^\alpha u + buE(|u|^\alpha))(t), \tag{2.9}$$

G as in (2.4). We want to use the Picard fixed-point theorem to find a solution of $u = \Phi(u)$ in

$$\overline{B(0, 3\delta_1)} = \{f \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1}) : \|f\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq 3\delta_1\}.$$

To prove $\Phi(\overline{B(0, 3\delta_1)} \subset \overline{B(0, 3\delta_1)}$, take $u \in \overline{B(0, 3\delta_1)}$. Using the hypothesis $\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \delta_1$ and Proposition 13 combined with the definition $\Phi(\cdot)$ in (2.9), we obtain

$$\begin{aligned}&\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq 2\Big(\delta_1 + \||u|^\alpha u\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + \|buE(|u|^\alpha)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\Big).\end{aligned}$$

Applying Proposition 11 and Holder's inequality we get

$$\begin{aligned}&\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq 2\Big(\delta_1 + \|u\|^{\alpha+1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + |b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u(t)\|^\alpha_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\Big).\end{aligned}$$

Using that $u \in \overline{B(0, 3\delta_1)}$ and choosing $0 < \delta_1 \ll 1$ we have

$$\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq 2c\delta_1 + 4c(3\delta_1)^{\alpha+1} + 4c|b|(3\delta_1)^{\alpha+1} < 3\delta_1.$$

Now we prove the contraction in $B(0,3\delta_1)$. Take $u,v\in B(0,3\delta_1)$:

$$\Phi(u)-\Phi(v)=iG(\chi(|v|^\alpha v-|u|^\alpha u))+iG\big(b(vE(|v|^\alpha)-uE(|u|^\alpha)\big).$$

By Proposition 13 we get

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq 2c\Big(\|v(|v|^\alpha-|u|^\alpha)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}+\||u|^\alpha(u-v)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\Big)\\ &\quad+2c|b|\Big(\|E(|v|^\alpha)(v-u)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\\ &\qquad+\|u(E(|v|^\alpha)-E(|u|^\alpha))\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\Big).\end{aligned}$$

Applying Holder's inequality and Proposition 11, we obtain

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq 2c\Big(\|v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\||v|^\alpha-|u|^\alpha\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\qquad+\|u\|^\alpha_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u-v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\Big)\\ &\qquad+2c|b|\Big(\|v\|^\alpha_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u-v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\qquad+\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\||v|^\alpha-|u|^\alpha\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\Big).\end{aligned}$$

Now we set $g(u)=|u|^\alpha$. It follows by the mean value theorem that

$$|g(u)-g(v)|\leq c(\alpha)(|u|^{\alpha-1}+|v|^{\alpha-1})|u-v|.$$

This property and Holder's inequality imply that

$$\begin{aligned}&\||v|^\alpha-|u|^\alpha\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq c(\alpha)\Big(\|u\|^{\alpha-1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u-v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\qquad+\|v\|^{\alpha-1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u-v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\Big).\end{aligned}$$

Finally by the last inequality and the hypothesis $u,v\in B(0,3\delta_1)$ we get

$$\|\Phi(u)-\Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\leq\delta_1^\alpha(c_1+c_2|b|)\|v-u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}.$$

Again taking $0 < \delta_1 \ll 1$ we get the contraction. □

Proof of Proposition 2. By hypothesis $u \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$. So by Holder's inequality and Proposition 11, we have, $|u|^\alpha u$ and $uE(|u|^\alpha)$ in $L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})$.

Now, by Remark 8 we can write

$$|u|^\alpha u = f_1 + f_2 \quad \text{and} \quad uE(|u|^\alpha) = f_3 + f_4, \tag{2.10}$$

where $f_j \in L^{p_j}(\mathbb{R}^{n+1})$ for some

$$1 \leq p_1 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_2 < \infty \;\; \text{and} \;\; 1 \leq p_3 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_4 < \infty.$$

Replacing (2.10) in (1.5) we get

$$u(t) = U(t)u_0 + i\chi G(f_1)(t) + i\chi G(f_2)(t) + ibG(f_3)(t) + ibG(f_4)(t). \tag{2.11}$$

Observe that from the decomposition (2.11) we have that $u(t) \in S'(\mathbb{R}^n)$.

Now, if we take $\phi \in S(\mathbb{R}^n)$ then $U(t)\phi \in C(\mathbb{R} : S(\mathbb{R}^n))$ and also $G(\phi)(t) \in C(\mathbb{R} : S(\mathbb{R}^n))$. By duality we can extend $U(t)$ to $S'(\mathbb{R}^n)$ and get $U(t)\phi \in C(\mathbb{R} : S'(\mathbb{R}^n))$ for $\phi \in S'(\mathbb{R}^n)$.

Using dominated convergence theorem we have $G(\phi)(t) \in C(\mathbb{R} : S'(\mathbb{R}^n))$ for $\phi \in S'(\mathbb{R}^n)$ and by (2.11)

$$u(t) \in C(\mathbb{R} : S'(\mathbb{R}^n)). \tag{2.12}$$

Letting $t \to 0$ in (2.11) we get $u(0) = u_0$.

Now we prove that $u(t)$ satisfies the equation

$$iu_t + \delta u_{x_1x_1} + \sum_{j=2}^{n} u_{x_jx_j} = \chi|u|^\alpha u + buE(|u|^\alpha), \tag{2.13}$$

in $S'(\mathbb{R}^n)$ for all $t \in \mathbb{R}$:

Define $F(u) := \chi|u|^\alpha u + buE(|u|^\alpha)$. Note that by (2.10) and (2.12) we have $F(u)(t) \in C(\mathbb{R}, S'(\mathbb{R}^n))$. Using the integral equation (1.5) and the definition of the operator G in (2.4) we have the following expression for $u(t)$

$$u(t) = U(t)u_0 + iG(Fu)(t). \tag{2.14}$$

Using group properties, Lebesgue dominated convergence theorem and the Lebesgue differentiation theorem combined with the expression of $u(t)$ in (2.14) we obtain that for any $\phi \in S(\mathbb{R}^n)$

$$i \lim_{h\to 0} \langle \frac{u(t+h) - u(t)}{h}, \phi \rangle = \langle -(\delta\partial_{x_1x_1} + \sum_{j=2}^{n} \partial_{x_jx_j})u(t) + F(u)(t), \phi \rangle,$$

where

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)dx,$$

which proves (2.13).

To prove $\|u(t_0)\|_Y < \infty$, take $r = \frac{\alpha(n+2)}{2}$ in inequality (2.7) of Proposition 13. Then we have

$$\|TT^*F\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq c\|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}. \tag{2.15}$$

From the last property and identity (6), $\forall\ t_0 \in \mathbb{R}$ we get

$$\|U(t)\int_{-\infty}^{+\infty} U(t_0 - s)F(s)ds\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}.$$

Now taking $\chi_{(0,t_0)}F$ instead of F in the last inequality we have

$$\|U(t)G(F)(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}, \tag{2.16}$$

where G was defined in (2.4).

Now taking $t = t_0$ in the integral equation (1.5) and applying $U(t)$ we have

$$U(t)u(t_0) = U(t + t_0)u_0 + iU(t)G(\chi|u|^\alpha u + buE(|u|^\alpha))(t_0).$$

Combining property (6), inequality (2.16) and the same arguments as in Theorem 1 we obtain

$$\begin{aligned}\|U(t)u(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} &\leq 2\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4\|u\|^{\alpha+1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &+ 4|b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u\|^{\alpha}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \infty.\end{aligned}$$

Finally, to prove the last statement of the theorem we set

$$u_+ = u_0 + i\int_0^\infty U(-\tau)(\chi|u|^\alpha u + buE(|u|^\alpha))(\tau)d\tau.$$

It follows from inequality (2.15) that

$$\begin{aligned}&\|U(t)u_+\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\\ &\leq 2\Big(\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|(\chi|u|^\alpha u + buE(|u|^\alpha))\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\Big) < \infty.\end{aligned}$$

We deduce from the decompositions in (2.10) that

$$U(-t)u(t) - u_+ = \int_t^\infty U(-\tau)(\chi|u|^\alpha u + buE(|u|^\alpha))(\tau)d\tau \to 0$$

in $S'(\mathbb{R}^n)$ as $t \to \infty$. The result for $t \to -\infty$ is proved similarly. □

3. Self-similar solutions

In this section we find self-similar solutions to (1.2) for $\delta > 0$. Without loss of generality we can suppose $\delta = 1$, so our equation becomes

$$\begin{cases} iu_t + \Delta u &= \chi|u|^\alpha u + buE(|u|^\alpha) \\ u(x,0) &= u_0(x) \end{cases} \quad \forall\, x \in \mathbb{R}^n,\ n = 2, 3,\ t \in \mathbb{R}. \tag{3.1}$$

We will need the following proposition:

Proposition 14. *Let $\varphi(x) = |x|^{-p}$ where $0 < \operatorname{Re} p < n$. Then $e^{it\Delta}\varphi$ is given by the explicit formula below for $x \neq 0$ and $t > 0$:*

$$\begin{aligned} e^{it\Delta}\varphi(x) =& |x|^{-p}\sum_{k=0}^{m} A_k(a,b)e^{k\pi i/2}\left(\frac{|x|^2}{4t}\right)^{-k} \\ &+ |x|^{-p}A_{m+1}(a,b)\left(\frac{|x|^2}{4t}\right)^{-m-1}\frac{(m+1)e^{aki/2}}{\Gamma(m+2-b)} \\ &\times \int_0^\infty\int_0^1 (1-s)^m\left(-i-\frac{4ts\tau}{|x|^2}\right)^{-a-m-1} e^{-\tau}\tau^{m+1-b}ds\,d\tau \\ &+ e^{i|x|^2/4t}|x|^{-n+p}(4t)^{\frac{n}{2}-p}\sum_{k=0}^{l} B_k(b,a)e^{-(n+2k)\pi i/4}\left(\frac{|x|^2}{4t}\right)^{-k} \\ &+ e^{i|x|^2/4t}|x|^{-n+p}(4t)^{\frac{n}{2}-p}B_{l+1}(b,a)\left(\frac{|x|^2}{4t}\right)^{-l-1}\frac{(l+1)e^{aki/2}}{\Gamma(l+2-b)} \\ &\times \int_0^\infty\int_0^1 (1-s)^l\left(-i-\frac{4ts\tau}{|x|^2}\right)^{-b-l-1} e^{-\tau}\tau^{l+1-a}ds\,d\tau, \end{aligned}$$

where $a = p/2$, $b = (n-p)/2$, $m, l \in \mathbb{N}$ such that $m + 2 > \operatorname{Re} b$ and $l + 2 > \operatorname{Re} a$, and

$$A_k(a,b) = \frac{\Gamma(a+k)\Gamma(k+1-b)}{\Gamma(a)\Gamma(1-b)k!},\ B_k(b,a) = \frac{\Gamma(b+k)\Gamma(k+1-a)}{\Gamma(a)\Gamma(1-a)k!},$$

where Γ denotes the gamma function.

Proof. We refer to [10] for a proof of this proposition. □

We already know that a self-similar solution must have a homogeneous initial condition with degree $-2/\alpha$. So the idea is to prove that $u_0(x) = \epsilon|x|^{-2/\alpha} \in Y$, where $0 < \epsilon \ll 1$. Then by Theorem 1 and Proposition 2 we have existence and uniqueness for equation (3.1) in Y. Since $u(x,t)$ and $\beta^{2/\alpha}u(\beta x, \beta^2 t)$ are both solutions, we must have $u = u_\beta$ and therefore self-similar solutions in Y.

To prove that $u_0 \in Y$, we consider the homogeneous problem with initial condition
$u_0(x) = |x|^{-2/\alpha}$:

$$\begin{cases} iu_t + \Delta u &= 0 \\ u(x,0) &= |x|^{-2/\alpha} \end{cases} \qquad \forall\, x \in \mathbb{R}^n,\ n = 2,3,\ t \in \mathbb{R}. \tag{3.2}$$

We know that the solution to the equation (3.2) is given by

$$u(x,t) = U(t)u_0(x),$$

where $U(t) = e^{it\Delta}$.

Since $u_\beta(x,t) = \beta^{2/\alpha}u(\beta x, \beta^2 t),\ \beta > 0$ is also a solution; we must have

$$\beta^{2/\alpha}u(\beta x, \beta^2 t) = U(t)u_0(x) = u(x,t).$$

Taking $\beta = 1/\sqrt{t}$ we get

$$u(x,t) = t^{-1/\alpha} f(x/\sqrt{t}), \tag{3.3}$$

where $f(x) = u(x,1)$.

By Proposition 14 we have that for $\alpha > 2/n$

$$|f(x)| \leq c(1+|x|)^{-\sigma} \text{ where } \sigma = \begin{cases} 2/\alpha; & \alpha \geq 4/n \\ n - 2/\alpha; & \alpha < 4/n. \end{cases} \tag{3.4}$$

Next, we calculate $\alpha(\lambda, u) = |\{(x,t) : |u(x,t)| > \lambda\}|$.
By (3.3) and (3.4)

$$\begin{aligned} \alpha(\lambda,u) &\leq \int_{\{(x,t):|t^{-1/\alpha}\left(1+\frac{|x|}{\sqrt{t}}\right)^{-\sigma}|>\lambda\}} d(x,t) \\ &\leq \int_{\{(x,t):0\leq t<\lambda^{-\alpha} \text{ and } |x|<t^{1/2}[(t\lambda^\alpha)^{-1/\alpha\sigma}-1]\}} d(x,t) \\ &\leq c\lambda^{-n/} \int_0^{\lambda^{-\alpha}} t^{\frac{n}{2}-\frac{n}{\sigma\alpha}}[1-(t\lambda^\alpha)^{\frac{1}{\sigma\alpha}}]^n dt \leq \lambda^{\frac{-\alpha(n+2)}{2}}. \end{aligned}$$

Therefore, $\|U(\cdot)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leq c$. Choosing $0 < \epsilon \ll 1$ and taking the initial condition $u_0(x) = \epsilon|x|^{-2/\alpha}$ we conclude the result.

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