



On the two-power nonlinear Schrödinger equation with non-local terms in Sobolev–Lorentz spaces

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Abstract. We are concerned with the two-power nonlinear Schrödinger-type equations with non-local terms. We consider the framework of Sobolev–Lorentz spaces which contain singular functions with infinite-energy. Our results include global existence, scattering and decay properties in this singular setting with fractional regularity index. Solutions can be physically realized because they have finite local L^2 -mass. Moreover, we analyze the asymptotic stability of solutions and, although the equation has no scaling, show the existence of a class of solutions asymptotically self-similar w.r.t. the scaling of the single-power NLS-equation. Our results extend and complement those of Weissler (Adv Differ Equ 6(4):419–440, 2001), particularly because we are working in the larger setting of Sobolev-weak- L^p spaces and considering non-local terms. The two nonlinearities of power-type and the generality of the non-local terms allow us to cover in a unified way a large number of dispersive equations and systems.

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1. Introduction

We consider the initial value problem (IVP) associated with the two-power nonlinear Schrödinger equation with non-local term

$$\begin{cases} i\partial_t u + Lu = a|u|^\alpha u + bE(|u|^\gamma)u, \\ u(x, 0) = u_0(x) \end{cases} \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1, \quad (1.1)$$

where $u = u(x, t)$ is a complex-valued function, a and b are complex constants, $0 < \alpha < \gamma$ are positive real numbers, E is a non-local linear operator, and L is a linear operator defined through its Fourier transform as

$$\widehat{Lu}(\xi) = q(\xi)\widehat{u}, \quad \xi \in \mathbb{R}^n.$$

Throughout the paper we assume the following:

(H1) the function q is real and homogeneous of degree d , that is,

$$q(\lambda\xi) = \lambda^d q(\xi), \quad \lambda > 0.$$

(H2) The function $G(x) = \int_{\mathbb{R}^n} e^{i(x\xi + q(\xi))} d\xi$ belongs to $L^\infty(\mathbb{R}^n)$.

(H3) The operator E is bounded in $L^{(p,\infty)}(\mathbb{R}^n)$, for $1 < p < \infty$, and commutes with fractional derivatives.

In **(H3)**, $L^{(p,\infty)}(\mathbb{R}^n)$ stands for the so-called weak- L^p spaces which, in view of Chebyshev's inequality, can be regarded as natural extensions of L^p -spaces. As will be seen later, conditions **(H1)**–**(H2)** are sufficient to prove dispersive-type estimates, which in turn are used to deal with the linear part of the associated integral equation. Also, **(H3)** is sufficient to handle with the nonlinear terms involving the operator E in our functional setting.

When L stands for the Laplacian operator and $b = 0$, the equation in (1.1) reduces to the well-known single-power nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = a|u|^\alpha u, \quad (1.2)$$

which appears in many physical situations. So, at a first glance, (1.1) can be seen as a mathematical extension of (1.2). However, several physical relevant models can be written in the form (1.1). Indeed, let us recall some examples. When $n = 2$, $L = m_1 \partial_{x_1}^2 + \partial_{x_2}^2$, $m_1 \in \mathbb{R} \setminus \{0\}$, and E is defined in Fourier variables as

$$\widehat{E(f)}(\xi) = \frac{\xi_1^2}{\xi_1^2 + m_2 \xi_2^2} \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2), \quad m_2 > 0, \quad (1.3)$$

(1.1) reduces to the so-called Davey-Stewartson (DS) system, which was derived in [16] (see also [24]) to model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave amplitude is modulated slowly in two horizontal directions. A generalized DS system describing the interaction of longwaves and shortwaves propagating in an infinite elastic medium (see [3]) can also be written in the form (1.1), where again $L = m_1 \partial_{x_1}^2 + \partial_{x_2}^2$, $m_1 \in \mathbb{R} \setminus \{0\}$, and E is defined by

$$\widehat{E(f)}(\xi) = \frac{\lambda \xi_1^4 + (1 + m_2 - 2\ell) \xi_1^2 \xi_2^2 + m_3 \xi_2^4}{(\xi_1^2 + m_2 \xi_2^2)(\lambda \xi_1^2 + m_3 \xi_2^2)} \widehat{f}(\xi), \quad (1.4)$$

with the real constants m_2, m_3, λ , and ℓ satisfying the relation

$$(\lambda - 1)(m_3 - m_2) = \ell^2.$$

An example in dimension $n = 3$ is given by the Shrira system (see [42]), which models the evolution of a three-dimensional packet of weakly nonlinear internal

gravity waves propagating obliquely at an arbitrary angle to the vertical. In this case,

$$L = \frac{\omega_{kk}}{2} \partial_{x_1}^2 + \frac{\omega_{ll}}{2} \partial_{x_2}^2 + \frac{\omega_{nn}}{2} \partial_{x_3}^2 + \omega_{nk} \partial_{x_1 x_3}^2$$

with $\omega_{kk} \neq 0$, $\omega_{nn} \neq 0$, $\omega_{ll}(\omega_{kk}\omega_{nn} - \omega_{nk}^2) \neq 0$, and E is given by

$$\widehat{E(f)}(\xi) = \nu \frac{\xi_2^2}{\xi_1^2 + \xi_2^2} \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3), \quad \nu \in \mathbb{R}. \quad (1.5)$$

Another three-dimensional example appears in the description of Bose-Einstein condensation of dipolar quantum gases (see [1, 36]). In such a case, L is the Laplacian operator and E is given by

$$\widehat{E(f)}(\xi) = \frac{4\pi}{3} \frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2} \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3). \quad (1.6)$$

For additional information concerning the above models and generalizations, we refer the reader to [2, 3, 14, 16, 21, 24–31, 42, 47]). Note, in particular, that the symbols of the operators E in (1.3), (1.4), and (1.6) are homogeneous of degree zero. Hence, one can apply the Calderon-Zygmund theory (see, for instance, [22]) to see that E is bounded from L^q to L^q , for any $1 < q < \infty$; so, after applying real interpolation we see that **(H3)** holds. For E defined as in (1.5) we cannot directly apply the Calderon-Zygmund theory. However, by interpolation with BMO and Hardy spaces we can still prove that E is bounded from L^q to L^q , for any $1 < q < \infty$ (see, [[25], page 184]), implying that **(H3)** also holds in this case. In addition, recalling that

$$\int_{\mathbb{R}} e^{i(a\eta^2 + y\eta)} d\eta = \sqrt{\frac{\pi}{|a|}} e^{-i\frac{y^2}{4a}} e^{i\frac{\pi}{4} \operatorname{sgn}(a)}, \quad a, y \in \mathbb{R},$$

it is easily seen that **(H2)** also holds in the above examples.

Notice also that we are able to consider higher-order operators L , as we take the parameter $d \geq 2$, such as polyharmonic operators. In particular, our results also apply to the fourth-order nonlinear Schrödinger equation

$$i\partial_t u + \mu \Delta^2 u = a|u|^\alpha u, \quad \mu \neq 0,$$

which was introduced in [32, 33] to take into account the role of fourth-order dispersion in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. To see that **(H2)** also holds here, we refer the reader to [5].

Equation (1.2) is one of most studied dispersive equation. To avoid too many references we cite, for instance, the books [8, 20, 37], where the reader will find a large class of results in different function spaces. We only emphasize that results on self-similar solutions, among others, were obtained, for instance, in [9–11, 41, 43]. Especially when L is the Laplacian and E is the identity operator I , (1.1) has been studied in recent years, to cite a few works, see [12, 38, 39, 44] and their references. The most disseminated results are obtained in the H^1 -setting, which provides finite energy solutions. The studied topics cover local and global well-posedness, scattering, radial symmetry and asymptotic behavior of solutions. However, in [47] solutions of (1.1) with $E = I$ were

obtained in the spaces $H^{s,q}(\mathbb{R}^n) = (I - \Delta)^{-s/2} L^q(\mathbb{R}^n)$, with $s > 0$ and $1 < q < \infty$, provided that the initial data satisfies some additional conditions.

For a general operator satisfying the assumption **(H3)**, existence and asymptotic properties in the single-power case $\gamma = \alpha$ were analyzed, in the context of weak Lebesgue spaces, in [4, 7] (see also [46]).

A substantial mathematical difference between (1.1) and (1.2) is the failure of scaling in (1.1); it is easily seen that the map

$$u(x, t) \mapsto u_\lambda(x, t) := \lambda^{2/\alpha} u(\lambda x, \lambda^2 t), \quad \lambda > 0 \quad (1.7)$$

leaves (1.2) invariant (i.e., if u is solution so is u_λ) but the same is not true for (1.1) (with $L = \Delta$ and $d = 2$).

Our main aim here is twofold: to provide a larger class for global existence and uniqueness by considering the framework of Sobolev-weak- L^p spaces with fractional regularity index $0 < s < 1$ (see Sect. 2 for the definitions), which contain singular functions with infinite-energy and allow to analyze self-similar asymptotics; and to consider non-local operators in the nonlinearities which allow to address in a unified way a number of dispersive models, including the above examples.

The aspects above, in turn, bring several additional difficulties. Thus we need to extend some well known results in the context of Lebesgue spaces to the spaces we are interested in, as well as to take into account the influence of the nonlocal operators in the estimates. We believe that those extended results have prospects to be applied in many other situations.

Also, we obtain scattering, decay properties of solutions and asymptotic stability results in that singular setting. In Remark 3.5, we point out that the data-solution map is Lipschitz continuous and then we have in fact a well-posedness result in the sense of Hadamard. Since weak- L^p spaces embed into L^2_{loc} for $p > 2$, solutions have finite local L^2 -mass and can be realized in physical space in any region with finite measure. In particular, all convergence and stability properties obtained here occur in the sense of L^2 -mass in any finite-volume region, no matter how large it is.

Asymptotic stability results for NLS type equations and systems are useful for characterizing solutions that, after initial perturbations, essentially recovers their profiles at large times. Asymptotic (or not) self-similarity type symmetries appear in physical situations and are used to describe phenomena in different spatial-temporal scales, revealing internal symmetry and structure in a system (see [17, 18]). For instance, in [19], they show that a type of self-similar parabolic pulse, called similaritons, is an asymptotic solution to a NLS-like equation with gain. Although the equation (1.1) has no scaling, we are able to prove the existence of a class of solutions asymptotically self-similar with respect to the scaling (1.7) of the equation $i\partial_t v + Lv = a|v|^\alpha v$, as $t \rightarrow \infty$. This means that, for large times, those solutions approximately preserve the self-similar structure of an important related model. Our self-similar asymptotics correspond to homogeneous initial data u_0 of degree $-d/\alpha$ and can be expressed as

$$v(x, t) = t^{-1/\alpha} V(xt^{-1/d}), \quad (1.8)$$

where the profiles V belong to some weak- L^p spaces. This provides another motivation in order to consider weak- L^p and Sobolev-weak- L^p spaces since they are the natural environment for homogeneous functions and allow the analysis of self-similar asymptotics and pulses as in [17, 18] and [19]. Moreover, the existence of self-similar asymptotics can also be used to study wave collapse (blow up solutions) by using the pseudo-conformal transformation (see [9]). This singular behavior has appeared in the context of nonlinear optics and been observed in numerical experiments for some Schrödinger-type equations and systems (see, e.g., [35]). We believe our solutions may be obtained from numerical methods because they are the limit of a Picard sequence come from a contraction argument and are L^2_{loc} -stable.

As usual, to study the IVP (1.1) we use its equivalent integral formulation

$$u(t) = U(t)u_0 + i \int_0^t U(t-s)(a|u|^\alpha u + bE(|u|^\gamma)u)(s)ds, \quad (1.9)$$

where $U(t)u_0$ is the solution of the linear problem

$$\begin{cases} i\partial_t u + Lu = 0, \\ u(x, 0) = u_0(x) \end{cases} \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

that is,

$$U(t)u_0(x) = \int_{\mathbb{R}^n} e^{i(x\xi + tq(\xi))} \widehat{u}_0(\xi) d\xi.$$

In view of our assumptions **(H1)** and **(H2)** the operator $U(t)$ defines a unitary group on $H^s(\mathbb{R}^n)$, for any $s \in \mathbb{R}$ (see [25]). Without loss of generality, from now on we shall consider $t \geq 0$ in (1.9) and in time-dependent functional spaces dealt with. The case $t \leq 0$ can be treated in a complete parallel way.

To simplify the calculations let us rewrite the integral equation (1.9) as

$$u(t) = U(t)u_0 + B(u),$$

where

$$B(u) = i \int_0^t U(t-s)(a|u|^\alpha u + bE(|u|^\gamma)u)(s)ds. \quad (1.10)$$

The paper is organized as follows. In Sect. 2 we introduce some notation, recall some results in Lebesgue spaces and prove their extensions to Sobolev–Lorentz spaces. These extensions (and interpolation) will play a key role in Sect. 3, where we prove our main results: global existence, scattering, decay properties, asymptotic stability, and existence of asymptotically self-similar solutions for the integral equation (1.9).

2. Notation and preliminaries

Let us begin this section by introducing the notation used throughout the paper. We use C or c to denote various constants that may vary line by line.

We denote by $\|\cdot\|_{L^p}$, $1 \leq p \leq \infty$, the usual Lebesgue L^p -norm. The Fourier transform of a function $f = f(x)$, is defined by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

The inverse Fourier transform of a function $g = g(\xi)$ is denoted by $(\mathcal{F}^{-1}g)(x) = \check{g}(x)$. In $\mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions) the Fourier transform is understood in the usual sense. $\mathcal{S}(\mathbb{R}^n)$ will denote the class of all Schwartz functions.

The standard Lorentz space is denoted by $L^{(p,q)}$, $0 < p, q \leq \infty$. In particular, (the weak Lebesgue spaces) $L^{(p,\infty)} = L^{(p,\infty)}(\mathbb{R}^n)$, $1 \leq p < \infty$, is defined as

$$L^{(p,\infty)} = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{C} \text{ measurable ; } \|f\|_{L^{(p,\infty)}}^* := \sup_{\lambda > 0} \lambda \alpha(\lambda, f)^{1/p} < \infty \right\},$$

where

$$\alpha(\lambda, f) = \mu(\{x \in \mathbb{R}^n; |f(x)| > \lambda\}), \text{ and } \mu \text{ is the Lebesgue measure.}$$

The quantity $\|\cdot\|_{L^{(p,\infty)}}^*$ is a quasi-norm in $L^{(p,\infty)}$. As is well-known, for $1 < p < \infty$ there exists an equivalent norm $\|\cdot\|_{L^{(p,\infty)}}$ in $L^{(p,\infty)}$ (i.e., it induces the same topology than $\|\cdot\|_{L^{(p,\infty)}}^*$), such that $L^{(p,\infty)}$ becomes a Banach space (see, for instance, Remark 1.4.12 in [22]). More precisely, we can define $\|\cdot\|_{L^{(p,\infty)}}$ as

$$\|f\|_{L^{(p,\infty)}} = \sup_{t > 0} t^{\frac{1}{p}} f^{**}(t)$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad \text{and} \quad f^*(t) = \inf\{\lambda > 0 : \alpha(\lambda, f) \leq t\}.$$

Moreover,

$$L^p \hookrightarrow L^{(p,\infty)}$$

with continuous embedding.

We observe that if $1 < p, q, r < \infty$, then the Hölder inequality

$$\|fg\|_{L^{(r,\infty)}} \leq \|f\|_{L^{(p,\infty)}} \|g\|_{L^{(q,\infty)}}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

holds (see [40]). Also, if $1 \leq r < \infty$ and $1 < q, p < \infty$, then the Young inequality

$$\|f * g\|_{L^{(q,\infty)}} \leq C \|g\|_{L^{(r,\infty)}} \|f\|_{L^{(p,\infty)}}, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \quad (2.1)$$

is valid (see also [22, pages 21 and 73]), where $L^{(r,\infty)}$ should be replaced by L^r when $r = 1$.

If (X_1, X_2) is a pair of compatible Banach spaces, $\theta \in (0, 1)$, and $1 \leq q \leq \infty$, let $(X_1, X_2)_{\theta, \infty}$ denote the interpolation space with respect to the couple (X_1, X_2) using the K -Method (see e.g., [6, 22], or [45]). We recall that $L^{(p,q)}$ can be defined as an interpolation between two Lebesgue spaces. More precisely, we have:

Theorem 2.1. (*Interpolation theorem*) Given $0 < p_0 < p_1 \leq \infty$, for any p, q and θ such that $p_0 < q \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < \theta < 1$ we have:

$$(L^{p_0}, L^{p_1})_{\theta, q} = L^{(p, q)} \quad \text{with} \quad \|f\|_{(L^{p_0}, L^{p_1})_{\theta, q}} = \|f\|_{L^{(p, q)}},$$

where, for $q < \infty$,

$$\begin{aligned} (L^{p_0}, L^{p_1})_{\theta, q} &= \left\{ a \text{ is Lebesgue measurable; } \|a\|_{(L^{p_0}, L^{p_1})_{\theta, q}} \right. \\ &\quad \left. := \left(\int_0^\infty t^{-\theta} k(t, a)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \end{aligned}$$

and

$$\begin{aligned} (L^{p_0}, L^{p_1})_{\theta, \infty} &= \left\{ a \text{ is Lebesgue measurable; } \|a\|_{(L^{p_0}, L^{p_1})_{\theta, \infty}} \right. \\ &\quad \left. := \sup_{t>0} t^{-\theta} k(t, a) < \infty \right\}, \end{aligned}$$

with

$$k(t, a) = \inf_{a=a_0+a_1} (\|a_0\|_{L^{p_0}} + t\|a_1\|_{L^{p_1}}).$$

Proof. We refer the reader to [6, Theorem 5.2.1]. $\square$

Remark 2.2. Theorem 2.1 also holds if we replace the spaces $L^{p_k} = L^{p_k}(\mathbb{R}^n)$ by $L^{p_k}(\mathcal{B}) = L^{p_k}(\mathbb{R}^n; \mathcal{B})$, the space of all measurable functions with values in the Banach space $\mathcal{B}$. In this case, $L^{p, q}(\mathcal{B}) = L^{p, q}(\mathbb{R}^n; \mathcal{B})$ is the interpolation space.

As usual, the operators J^{-s} and Λ^{-s} will denote the Bessel and Riesz potentials of order s , thus

$$J^s f(x) = \{(1 + |\xi|^2)^{s/2} \widehat{f}\}^\vee(x)$$

and

$$\Lambda^s f(x) = (|\xi|^s \widehat{f})^\vee(x).$$

The base space we are interested in is presented next. We define the homogeneous Sobolev–Lorentz space $\dot{H}_{p, \infty}^s$ to be the set of all tempered distributions f such that $\Lambda^s f$ belongs to $L^{(p, \infty)}$, that is,

$$\dot{H}_{p, \infty}^s := \{f \in \mathcal{S}'; \|\Lambda^s f\|_{L^{(p, \infty)}} < \infty\}.$$

The inhomogeneous space $H_{p, \infty}^s$ is defined in a similar fashion by setting

$$H_{p, \infty}^s := \{f \in \mathcal{S}'; \|J^s f\|_{L^{(p, \infty)}} < \infty\}.$$

Let us recall the Littlewood–Paley theory: let $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^n)$ be a function satisfying $0 \leq \widehat{\varphi} \leq 1$, $\widehat{\varphi} = 1$ if $|\xi| \leq 1$, and $\widehat{\varphi} = 0$ if $|\xi| > 2$. Define

$$\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi), \quad \widehat{\psi}_j(\xi) = \widehat{\psi}(2^{-j}\xi), \quad j \in \mathbb{Z},$$

so that

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}_j(\xi) = 1, \quad \xi \neq 0, \quad \text{and} \quad \text{supp}(\widehat{\psi}_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}.$$

The Littlewood–Paley multipliers Δ_j are defined as

$$\Delta_j f = (\widehat{\psi}_j \widehat{f})^\vee = \psi_j * f, \quad j \in \mathbb{Z}. \quad (2.2)$$

Also, let $\widehat{\eta}$ be another smooth function supported in $\{1/4 < |\xi| < 4\}$ such that $\widehat{\eta} = 1$ on $\text{supp}(\widehat{\psi})$. We define $\widetilde{\Delta}_j$ like Δ_j with η instead of ψ . Thus, the identity

$$\widetilde{\Delta}_j \Delta_j = \Delta_j \quad (2.3)$$

holds. With this notation in hand, the *Littlewood–Paley function* defines an equivalent norm in the Lebesgue spaces. That is to say:

Theorem 2.3. (Littlewood–Paley) *Let $1 < p < \infty$. Then there exist positive constants c_p and C_p such that, for any $f \in L^p(\mathbb{R}^n)$,*

$$c_p \|f\|_p \leq \left\| \left(\sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p.$$

Proof. See Theorem 5.1.2 in [22]. $\square$

Next, we recall that the Hardy–Littlewood maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r)} \int_{B_r} |f(x-y)| dy. \quad (2.4)$$

Here f is any locally integrable function and B_r is the Euclidean ball of radius r centered at the origin. A well known property of M is given below.

Theorem 2.4. *The operator M is weak-type $(1, 1)$ and strong-type (p, p) , $1 < p \leq \infty$.*

Proof. See Theorem 2.1.6 in [22]. $\square$

In what follows we denote by $L^p(l^2)$, $1 < p < \infty$, the space of all sequences (f_k) of measurable functions on $\mathbb{R}^n$ satisfying

$$\|(f_k)\|_{L^p(l^2)} = \left\| \left(\sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \infty.$$

Given a linear operator T acting on $L^p(\mathbb{R}^n)$ and taking values in the set of measurable functions, we can define an l^2 -valued extension, said as $\vec{T}$, by setting

$$\vec{T}(\{f_k\}_k) := \{Tf_k\}_k.$$

Lemma 2.5. *If T is a bounded linear operator from L^p to L^q , $1 < p, q < \infty$, then $\vec{T}$ is also a bounded linear operator from $L^p(l^2)$ to $L^q(l^2)$.*

Proof. See Theorem 4.5.1 in [22]. $\square$

As an immediate consequence of Lemma 2.5 and Theorem 2.4 the Hardy–Littlewood maximal function has a bounded l^2 -valued extension (See Example 9.5.9 in [23]). More precisely.

Corollary 2.6. *If $\vec{M}$ denotes the l^2 -valued extension of M then, for $1 < p < \infty$,*

$$\|\vec{M}(\{f_k\})\|_{L^p(l^2)} \leq C_p \|\{f_k\}\|_{L^p(l^2)}, \quad (2.5)$$

for some constant $C_p > 0$.

By using the real interpolation method (see, for instance, [6] or [22] for details) we will extend the previous results to weak Lebesgue spaces. We start with the Littlewood–Paley inequality.

Lemma 2.7. *Let $1 < p < \infty$. Then there exist positive constants c_p and C_p such that, for any $f \in L^{(p,\infty)}(\mathbb{R}^n)$,*

$$c_p \|f\|_{L^{(p,\infty)}} \leq \left\| \left(\sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^{(p,\infty)}} \leq C_p \|f\|_{L^{(p,\infty)}}. \quad (2.6)$$

Proof. We will present two different proofs based on the real interpolation method.

First Proof. Let us begin by proving the second inequality. Define the following linear operator

$$\begin{aligned} T : L^p(\mathbb{R}^n) &\rightarrow L^p(l^2) \\ T(f) &= (\Delta_j f)_j. \end{aligned} \quad (2.7)$$

From Theorem 2.3 we know that T is well-defined. Now, taking $1 < p_0 < p_1 < \infty$ we also have from Theorem 2.3 that T is bounded from $L^{p_k}(\mathbb{R}^n)$ to $L^{p_k}(l^2)$, $k = 0, 1$. Applying the real interpolation method we obtain that T is also bounded from $(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta, \infty}$ to $(L^{p_0}(l^2), L^{p_1}(l^2))_{\theta, \infty}$ where $0 < \theta < 1$. Since

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta, \infty} = L^{(p,\infty)}(\mathbb{R}^n)$$

and

$$(L^{p_0}(l^2), L^{p_1}(l^2))_{\theta, \infty} = L^{(p,\infty)}(l^2),$$

where $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, we get the result.

To prove the first inequality we define, for each $1 < p < \infty$, the following subspace of $L^p(l^2)$:

$$A_p = \{\tilde{f} \in L^p(l^2); \exists f \in L^p(\mathbb{R}^n) \text{ s.t. } \tilde{f} = (\Delta_j f)_j\}.$$

From Theorem 2.3 we know that $(A_p, \|\cdot\|_{L^p(l^2)})$ is a Banach space. Now consider the linear operator T_1 defined by

$$\begin{aligned} T_1 : A_p &\rightarrow L^p(\mathbb{R}^n) \\ T_1(\tilde{f}) &:= \sum_j \Delta_j f = f. \end{aligned}$$

Using the same idea as before we take $1 < p_0 < p_1 < \infty$. From Theorem 2.3 we obtain that T_1 is bounded from A_{p_k} to $L^{p_k}(\mathbb{R}^n)$, $k = 0, 1$. From real interpolation we have that T_1 is bounded from $(A_{p_0}, A_{p_1})_{\theta, \infty}$ to $(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta, \infty}$. Thus, it suffices to prove that $(A_{p_0}, A_{p_1})_{\theta, \infty} = A_{p, \infty}$, where

$$A_{p, \infty} = \{\tilde{f} \in L^{(p, \infty)}(l^2); \exists f \in L^{(p, \infty)}(\mathbb{R}^n) \text{ s.t. } \tilde{f} = (\Delta_j f)_j\}.$$

Let $\tilde{f} \in A_{p, \infty}$. Then $\tilde{f} = (\Delta_j f)_j$ for some $f \in L^{(p, \infty)}(\mathbb{R}^n)$. Therefore,

$$\|\tilde{f}\|_{(A_{p_0}, A_{p_1})_{\theta, \infty}} = \sup_{t>0} t^{-\theta} \inf_{\tilde{f}=\tilde{f}_0+\tilde{f}_1} (\|\tilde{f}_0\|_{L^{p_0}(l^2)} + \|\tilde{f}_1\|_{L^{p_1}(l^2)}).$$

Since $\tilde{f}_0 \in A_{p_0}$ and $\tilde{f}_1 \in A_{p_1}$,

$$\tilde{f}_0 = (\Delta_j f_0)_j, \text{ for some } f_0 \in L^{p_0}(\mathbb{R}^n)$$

and

$$\tilde{f}_1 = (\Delta_j f_1)_j, \text{ for some } f_1 \in L^{p_1}(\mathbb{R}^n).$$

So,

$$\begin{aligned} \|\tilde{f}\|_{(A_{p_0}, A_{p_1})_{\theta, \infty}} &= \sup_{t>0} t^{-\theta} \inf_{(\Delta_j f)_j = (\Delta_j f_0)_j + (\Delta_j f_1)_j} (\|(\Delta_j f_0)_j\|_{L^{p_0}(l^2)} + \|(\Delta_j f_1)_j\|_{L^{p_1}(l^2)}) \end{aligned}$$

and, in view of Theorem 2.3,

$$\begin{aligned} \|\tilde{f}\|_{(A_{p_0}, A_{p_1})_{\theta, \infty}} &\leq C \sup_{t>0} t^{-\theta} \inf_{f=f_0+f_1} (\|f_0\|_{L^{p_0}(\mathbb{R}^n)} + \|f_1\|_{L^{p_1}(\mathbb{R}^n)}) \\ &= C \|f\|_{L^{(p, \infty)}(\mathbb{R}^n)} < \infty. \end{aligned}$$

On the other hand, let $\tilde{f} \in (A_{p_0}, A_{p_1})_{\theta, \infty}$. Then $\tilde{f} = \tilde{f}_0 + \tilde{f}_1$, with $\tilde{f}_0 \in A_{p_0}$ and $\tilde{f}_1 \in A_{p_1}$. We will show that $\tilde{f} \in A_{p, \infty}$. In fact,

$$\|\tilde{f}\|_{A_{p, \infty}} = \sup_{t>0} t^{-\theta} \inf_{\tilde{f}=F_0+F_1} (\|F_0\|_{L^{p_0}(l^2)} + \|F_1\|_{L^{p_1}(l^2)}).$$

Since $A_{p_0} \subset L^{p_0}(l^2)$ and $A_{p_1} \subset L^{p_1}(l^2)$ we conclude that

$$\|\tilde{f}\|_{A_{p, \infty}} \leq C \sup_{t>0} t^{-\theta} \inf_{\tilde{f}=\tilde{f}_0+\tilde{f}_1} (\|\tilde{f}_0\|_{L^{p_0}(l^2)} + \|\tilde{f}_1\|_{L^{p_1}(l^2)}),$$

where $\tilde{f}_0 \in A_{p_0}$ and $\tilde{f}_1 \in A_{p_1}$. Therefore

$$\|\tilde{f}\|_{A_{p, \infty}} \leq C \|\tilde{f}\|_{(A_{p_0}, A_{p_1})_{\theta, \infty}} < \infty,$$

which shows the desired result.

Second Proof. Define the norms

$$\|f\|_{L^p} = \left\| \left(\sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p} \quad \text{and} \quad \|f\|_{L^{(p, \infty)}} = \left\| \left(\sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^{(p, \infty)}}.$$

From Theorem 2.3, $\|\cdot\|_{L^p}$ is an equivalent norm in L^p , $1 < p < \infty$. This means that the identity operator $I : (L^{p_k}, \|\cdot\|_{L^{p_k}}) \rightarrow (L^{p_k}, \|\cdot\|_{L^{p_k}})$ is continuous for any $1 < p_1 < p_2 < \infty$. Hence, by real interpolation, I also is continuous from $(L^{(p, \infty)}, \|\cdot\|_{L^{(p, \infty)}})$ to $(L^{(p, \infty)}, \|\cdot\|_{L^{(p, \infty)}})$, $p_1 < p < p_2$, which means that the first inequality in (2.6) holds. Here, to see that the interpolation space between

$(L^{p_1}, \|\cdot\|_{L^{p_1}})$ and $(L^{p_2}, \|\cdot\|_{L^{p_2}})$ is indeed $(L^{(p,\infty)}, \|\cdot\|_{L^{(p,\infty)}})$ it is sufficient to recall that being L^p a retract of $L^p(l^2)$, $(L^{p_1}, L^{p_2})_{\theta,\infty}$ is also a retract of $(L^{p_1}(l^2), L^{p_2}(l^2))_{\theta,\infty}$ (see Theorems 6.4.2 and 6.4.3 in [6]).

The second inequality is obtained in a similar fashion, which concludes the proof. $\square$

Remark 2.8. In view of Lemma 2.7 one can see that the expression

$$\|\{2^{js}\Delta_j f\}\|_{L^{(p,\infty)}(l^2)} = \left\| \left(\sum_j 2^{2js} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^{(p,\infty)}}$$

defines an equivalent norm in $\dot{H}_{p,\infty}^s$. Indeed, note that $2^{js}\Delta_j f = \Delta_j^\sigma f_s$, where $f_s = \Lambda^s f$ and Δ_j^σ is the “Littlewood–Paley multiplier” given by $\widehat{\Delta_j^\sigma f}(\xi) = \widehat{\sigma(2^{-j}\xi)}\widehat{f}$, with $\widehat{\sigma}(\xi) = |\xi|^{-s}\widehat{\psi_j}(\xi)$. Thus applying Lemma 2.7 with Δ_j^σ instead of Δ_j , we obtain

$$\|\{2^{js}\Delta_j f\}\|_{L^{(p,\infty)}(l^2)} = \|\{\Delta_j^\sigma f_s\}\|_{L^{(p,\infty)}(l^2)} \sim \|f_s\|_{L^{(p,\infty)}}.$$

The interested reader will find the details (for L^p instead of $L^{(p,\infty)}$) in [23, Theorem 6.2.7]. In particular, this shows that the space $\dot{H}_{p,\infty}^s$ can be obtained as an interpolation space between two Sobolev spaces. More precisely

$$\dot{H}_{p,\infty}^s = (\dot{H}_{p_0}^s, \dot{H}_{p_1}^s)_{\theta,\infty}, \quad p_0 \neq p_1, \quad 0 < \theta < 1,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (see Theorem 1 in [45, page 184]).

In what follows, the space $L^{(p,\infty)}(l^2)$ is defined similarly to $L^p(l^2)$ when replacing $\|\cdot\|_{L^p}$ by $\|\cdot\|_{L^{(p,\infty)}}$.

Lemma 2.9. *If $\vec{M}$ denotes the l^2 -valued extension of M then, for $1 < p < \infty$, and $\{f_k\} \in L^{(p,\infty)}(l^2)$,*

$$\|\vec{M}(\{f_k\})\|_{L^{(p,\infty)}(l^2)} \leq C_p \|\{f_k\}\|_{L^{(p,\infty)}(l^2)}.$$

Proof. It suffices to follow the ideas in the proof of the previous lemma. From inequality (2.5) we have that $\vec{M}$ is bounded from $L^{p_0}(l^2)$ to $L^{p_0}(l^2)$ and from $L^{p_1}(l^2)$ to $L^{p_1}(l^2)$, where $1 < p_0 < p_1 < \infty$. From real interpolation we get the result. $\square$

Next result is an adapted version to weak Lebesgue spaces of the Sobolev embedding. Since $f(x) = \{|\xi|^{-s}\widehat{f_s}\}^\vee(x) = I_s(f_s)(x)$, where $f_s(x) = \Lambda^s f(x)$ and $I_s = \Lambda^{-s} = |x|^{-(n-s)}*$, it can be proved by using Young’s inequality for convolution operators in Lorentz spaces (see (2.1) and [22, page 73]).

Lemma 2.10. *Let s be a real number with $0 < s < n$ and let $1 \leq p < q < \infty$ satisfy*

$$s = \frac{n}{p} - \frac{n}{q}. \quad (2.8)$$

Then there exists a positive constant $C = C(n, s, p)$ such that, for any $f \in \dot{H}_{p,\infty}^s$,

$$\|f\|_{L^{(q,\infty)}} \leq C \|\Lambda^s f\|_{L^{(p,\infty)}}. \quad (2.9)$$

Remark 2.11. As a consequence of the previous lemma we obtain that

$$\|u\|_{L(\frac{\gamma(\alpha+2)}{\alpha}, \infty)} \leq C \|\Lambda^s u\|_{L(\alpha+2, \infty)},$$

for $0 < \alpha < \gamma$ and $s = \frac{n(\gamma-\alpha)}{\gamma(\alpha+2)}$.

In order to estimate the nonlinear part of our problem we also need the following lemma.

Lemma 2.12. *Let $0 < \alpha < \gamma$ and suppose that s defined by $s = \frac{n(\gamma-\alpha)}{\gamma(\alpha+2)}$ satisfies $0 < s < 1$. Then*

$$\|\Lambda^s(|u|^\alpha u)\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} \leq C \|u\|_{L(\alpha+2, \infty)}^\alpha \|\Lambda^s u\|_{L(\alpha+2, \infty)} \quad (2.10)$$

and

$$\|\Lambda^s(|u|^\gamma u)\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} \leq C \|\Lambda^s u\|_{L(\alpha+2, \infty)}^{\gamma+1}. \quad (2.11)$$

To prove Lemma 2.12 we need two additional results. The first one is the Leibniz rule for weak Lebesgue spaces.

Lemma 2.13. *Suppose that $F \in C^1(\mathbb{C}, \mathbb{C})$, $s \in (0, 1)$, and assume that $1 < p, q, r < \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then,*

$$\|\Lambda^s F(u)\|_{L(r, \infty)} \leq C \|F'(u)\|_{L(p, \infty)} \|\Lambda^s u\|_{L(q, \infty)}, \quad (2.12)$$

as long as the right-hand side is finite.

Proof. Since the ideas are the same as those in [13] we will only outline the proof making the necessary adaptations to Lorentz spaces.

At first we observe that

$$\Delta_j F(u)(x) = \int_{\mathbb{R}^n} \left[\int_0^1 F'(tu(y) + (1-t)u(x)) dt \right] (u(y) - u(x)) \psi_j(x-y) dy,$$

where Δ_j and ψ_j were defined in (2.2). Using that

$$\left| \int_0^1 F'(tu(y) + (1-t)u(x)) dt \right| \leq 2M(F'(u))(x)$$

and decomposing $u = \sum_k \tilde{\Delta}_k \Delta_k u$, we obtain

$$|\Delta_j F(u)(x)| \leq CM(F'(u))(x) \cdot \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |\tilde{\Delta}_k \Delta_k u(y) - \tilde{\Delta}_k \Delta_k u(x)| |\psi_j(x-y)| dy, \quad (2.13)$$

where M is the Hardy–Littlewood maximal function. Now we break the sum over k into the cases $k < j$ and $k \geq j$. By using the properties of ψ_j and Δ_j , we see that

$$\sum_{k < j} \int_{\mathbb{R}^n} |\tilde{\Delta}_k \Delta_k u(y) - \tilde{\Delta}_k \Delta_k u(x)| |\psi_j(x-y)| dy \leq C \sum_{k < j} 2^{k-j} M^2 \Delta_k u(x), \quad (2.14)$$

where $M^2(\Delta_k u(x)) = M \circ M(\Delta_k u(x))$. In a similar fashion,

$$\sum_{k \geq j} \int_{\mathbb{R}^n} |\tilde{\Delta}_k \Delta_k u(y) - \tilde{\Delta}_k \Delta_k u(x)| |\psi_j(x-y)| dy \leq C \sum_{k \geq j} M^2 \Delta_k u(x). \quad (2.15)$$

Inserting (2.14) and (2.15) into (2.13), substituting $j = k - m$ and applying Minkowski's inequality, we have

$$\begin{aligned} & \left(\sum_{j=-\infty}^{\infty} 2^{2js} |\Delta_j F(u)(x)|^2 \right)^{1/2} \\ & \leq CMF'(u)(x) \sum_{m=-\infty}^{\infty} 2^{-\epsilon|m|} \left(\sum_{k=-\infty}^{\infty} 2^{2ks} |M^2 \Delta_k u(x)|^2 \right)^{1/2}, \end{aligned}$$

where $\epsilon = 2 \min(s, 1 - s) > 0$.

In view of Remark 2.8, Holder's inequality and Lemma 2.9, we then deduce

$$\begin{aligned} \|\Lambda^s F(u)\|_{L(r,\infty)} & \leq C \|MF'(u)\|_{L(p,\infty)} \left\| \left(\sum_{k=-\infty}^{\infty} 2^{2ks} |M^2 \Delta_k u|^2 \right)^{1/2} \right\|_{L(q,\infty)} \\ & = C \|MF'(u)\|_{L(p,\infty)} \|\vec{M}^2 \{2^{ks} \Delta_j u\}\|_{L(q,\infty)(l^2)} \\ & \leq C \|MF'(u)\|_{L(p,\infty)} \|\{2^{ks} \Delta_j u\}\|_{L(q,\infty)(l^2)}. \end{aligned}$$

The conclusion now follows as an application of Remark 2.8. $\square$

Next Lemma is similar to Lemma A.2 in [34].

Lemma 2.14. *Suppose that $F \in C^1(\mathbb{C}, \mathbb{C})$ satisfies $F(0) = 0$ and $|F'(x)| \leq c|x|^{k-1}$, $k \geq 1$. If $s \in [0, 1]$, then*

$$\|\Lambda^s F(u)\|_{L(r,\infty)} \leq c \|u\|_{L(p,\infty)}^{k-1} \|\Lambda^s u\|_{L(q,\infty)} \quad (2.16)$$

where $1 < p, q, r < \infty$, $\frac{1}{r} = \frac{k-1}{p} + \frac{1}{q}$ and the constant c depends on s, p, q, r .

Proof. Let us start with the case $s = 0$. Since $F \in C^1(\mathbb{C}, \mathbb{C})$ and $F(0) = 0$ we have that

$$\left| \frac{F(u) - F(0)}{u - 0} \right| \leq c|u|^{k-1}.$$

Now from Holder's inequality,

$$\|F(u)\|_{L(r,\infty)} \leq c \|u\|_{L(\tilde{p}(k-1),\infty)}^{k-1} \|u\|_{L(q,\infty)} \quad \text{where} \quad \frac{1}{r} = \frac{1}{\tilde{p}} + \frac{1}{q}.$$

Taking $\tilde{p}(k-1) = p$ we get the result. To solve the case $s = 1$ we use the same idea together with the facts that

$$\partial F(u) = F'(u) \partial u \quad \text{and} \quad \|\partial u\|_{L(p,\infty)} = \|\Lambda u\|_{L(p,\infty)}.$$

Assume now $0 < s < 1$. From Lemma 2.13, we get

$$\|\Lambda^s F(u)\|_{L(r,\infty)} \leq c \|F'(u)\|_{L(\tilde{p},\infty)} \|\Lambda^s u\|_{L(q,\infty)} \quad \text{for} \quad \frac{1}{r} = \frac{1}{\tilde{p}} + \frac{1}{q}.$$

Using the hypothesis on F and Holder's inequality, we obtain

$$\|\Lambda^s F(u)\|_{L(r,\infty)} \leq c \|u\|_{L(\tilde{p}(k-1),\infty)}^{k-1} \|\Lambda^s u\|_{L(q,\infty)}.$$

By taking $\tilde{p}(k-1) = p$ we get the desired inequality. $\square$

With Lemma 2.14 in hand we are able to prove Lemma 2.12.

Proof of Lemma 2.12. To prove (2.10) we only need to choose $F(x) = |x|^\alpha x$, $r = \frac{\alpha+2}{\alpha+1}$, $p = q = \alpha + 2$, $k - 1 = \alpha$ and apply Lemma 2.14. To prove (2.11) we first note that, as above, an application of Lemma 2.13 gives,

$$\|\Lambda^s(|u|^\gamma u)\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} \leq C \|u\|_{L(\frac{\gamma(\alpha+2)}{\alpha}, \infty)}^\gamma \|\Lambda^s u\|_{L(\alpha+2, \infty)}.$$

An application of Remark 2.11 establishes the desired inequality. $\square$

Remark 2.15. In [34] it was proved (in the L^p level) that (2.16) also holds if $F \in C^m(\mathbb{C}, \mathbb{C})$ satisfies $|D^i F(x)| \leq |x|^{k-i}$, $i = 1, \dots, m$ for some $k \geq m$, and $s \in [0, m]$. The proof relies on the Gagliardo-Nirenberg type inequality

$$\|\Lambda^s f\|_{L^p} \leq C \|\Lambda^{s_0} f\|_{L^{p_0}}^{1-\theta} C \|\Lambda^{s_1} f\|_{L^{p_1}}^\theta, \quad (2.17)$$

where $\theta \in (0, 1)$, $s = (1 - \theta)s_0 + \theta s_1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Here, since we do not know if (2.17) holds in the $L^{(p, \infty)}$ level, we are unable to prove a similar result. The drawback is that we will not reach all ranges of α and γ as in [47].

Finally, next lemma establishes the boundedness of the linear group $U(t)$ in the weak Lebesgue spaces.

Lemma 2.16. *Let $1 < p < 2$. If p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$, then there exists a constant $C = C(n, p) > 0$ such that*

$$\|U(t)\phi\|_{L^{(p', \infty)}} \leq C t^{-\frac{n}{d}(\frac{2}{p}-1)} \|\phi\|_{L^{(p, \infty)}}, \quad (2.18)$$

for all $\phi \in L^{(p, \infty)}(\mathbb{R}^n)$ and $t > 0$.

Proof. We refer the reader to [7] (see also [43]) for a proof of this lemma. $\square$

3. Main results

In this section we will state and prove our main results. We follow the ideas in [47] where the author proves existence of global solutions for small initial data with respect to a norm which is related to the structure of the two-power nonlinear Schrödinger equation. Our results extend the ones in [47] since weak- L^p spaces contain Lebesgue's spaces. At first let us define the function spaces where the solutions will be obtained.

Definition 3.1. Given positive numbers s, β, δ and M , let $X_M = X_M(s, \beta, \delta)$ be the set of Bochner-measurable functions $u : (0, +\infty) \rightarrow H_{\alpha+2, \infty}^s$ such that

$$\|u\|_\beta := \sup_{t>0} t^\beta \|u(t)\|_{L(\alpha+2, \infty)} \leq M, \quad (3.1)$$

and

$$\|u\|_{\delta, s} := \sup_{t>0} t^\delta \|\Lambda^s u(t)\|_{L(\alpha+2, \infty)} \leq M. \quad (3.2)$$

It is not difficult to see that (X_M, d) is a nonempty complete metric space endowed with the distance

$$d(u, v) := \sup_{t>0} t^\beta \|u(t) - v(t)\|_{L^{(\alpha+2, \infty)}}.$$

For $\rho > 0$, we also define the initial data class $\mathcal{I}_\rho = \mathcal{I}_\rho(s, \beta, \delta)$ as follows

$$\mathcal{I}_\rho = \{u_0 \in \mathcal{S}'; U(t)u_0 \in X_\rho(s, \beta, \delta)\}. \quad (3.3)$$

Our first result reads as follows.

Theorem 3.2. (Global existence and uniqueness) *Assume $0 < \max\{1, \alpha\} < \gamma$. Define*

$$s = \frac{n(\gamma - \alpha)}{\gamma(\alpha + 2)} \quad (3.4)$$

and suppose that $0 < s < 1$ and

$$\frac{\alpha + 2}{\alpha + 1} < \frac{n\alpha}{d} < \alpha + 2. \quad (3.5)$$

Consider the positive numbers β and δ defined by

$$\beta = \frac{1}{\alpha} - \frac{n}{d(\alpha + 2)} \quad \text{and} \quad \delta = \frac{1}{\gamma} + \frac{s}{d} - \frac{n}{d(\alpha + 2)}. \quad (3.6)$$

Let $\rho > 0$ and $M > 0$ be such that

$$\rho + |a|C_1M^{\alpha+1} + |b|C_2M^{\gamma+1} \leq M \quad (3.7)$$

and

$$\rho + |a|C_3M^{\alpha+1} + |b|C_4M^{\gamma+1} \leq M \quad (3.8)$$

for some positive constants C_1, C_2, C_3, C_4 given in the calculations below, and assume that

$$|a|C_1M^\alpha + |b|C_2M^\gamma < 1. \quad (3.9)$$

If $u_0 \in \mathcal{I}_\rho$, then there exists a unique global solution of (1.9), say, $u \in X_M(s, \beta, \delta)$.

Proof. The proof is based on the Banach fixed point theorem. We consider the integral operator

$$(\Phi u)(t) = U(t)u_0 + (Bu)(t), \quad (3.10)$$

where B is defined as in (1.10).

Let $X_M = X_M(s, \beta, \delta)$ be the function space from Definition 3.1. We will show that Φ maps X_M into itself and $\Phi : X_M \rightarrow X_M$ is a contraction. To do that, we assume that $u, v \in X_M$ and estimate the integrals below:

$$\begin{aligned} J_1 &:= t^\beta |a| \int_0^t \|U(t-\tau)(|u|^\alpha u - |v|^\alpha v)(\tau)\|_{L^{(\alpha+2, \infty)}} d\tau, \\ J_2 &:= t^\beta |b| \int_0^t \|U(t-\tau)[E(|u|^\gamma)u - E(|v|^\gamma)v](\tau)\|_{L^{(\alpha+2, \infty)}} d\tau, \\ J_3 &:= t^\delta |a| \int_0^t \|\Lambda^s U(t-\tau)(|u|^\alpha u)(\tau)\|_{L^{(\alpha+2, \infty)}} d\tau, \end{aligned}$$

and

$$J_4 := t^\delta |b| \int_0^t \|\Lambda^s U(t-\tau)[E(|u|^\gamma)u](\tau)\|_{L^{(\alpha+2,\infty)}} d\tau.$$

For $m > 0$, recall the pointwise inequality

$$||u|^m u - |v|^m v| \leq C(|u|^m + |v|^m)|u - v|. \quad (3.11)$$

To estimate J_1 , we use (3.11) with $m = \alpha$, Lemma 2.16 and Holder's inequality to get

$$\begin{aligned} J_1 &\leq |a| C t^\beta \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{2(\alpha+1)}{\alpha+2}-1)} \|(|u|^\alpha u - |v|^\alpha v)(\tau)\|_{L^{(\frac{\alpha+2}{\alpha+1},\infty)}} d\tau \\ &\leq |a| C t^\beta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} (\|u(\tau)\|_{L^{(\alpha+2,\infty)}}^\alpha \\ &\quad + \|v(\tau)\|_{L^{(\alpha+2,\infty)}}^\alpha) \|u(\tau) - v(\tau)\|_{L^{(\alpha+2,\infty)}} d\tau \\ &\leq C |a| M^\alpha d(u,v) t^{1-\frac{n}{d}\frac{\alpha}{\alpha+2}-\alpha\beta} \int_0^1 (1-s)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-(\alpha+1)\beta} ds \\ &\leq |a| C_1 M^\alpha d(u,v), \end{aligned}$$

where in the last inequality we used that $\beta(\alpha+1) < 1$, $\frac{n\alpha}{d(\alpha+2)} < 1$, and

$$\beta\alpha + \frac{n\alpha}{d(\alpha+2)} = 1. \quad (3.12)$$

Now let us estimate J_2 . Firstly, we write

$$E(|u|^\gamma)u - E(|v|^\gamma)v = E(|u|^\gamma)(u-v) + E(|u|^\gamma - |v|^\gamma)v. \quad (3.13)$$

Let $z = \frac{\alpha+2}{\alpha}\gamma$ and $l = \frac{(\alpha+2)\gamma}{(\alpha+1)(\gamma-1)+1}$. Observe that $z, l > 1$, $\frac{1}{l} = \frac{1}{\alpha+2} + \frac{\gamma-1}{z}$ and $\frac{1}{l} + \frac{1}{z} = \frac{\alpha+1}{\alpha+2}$. Then, using Holder's inequality, assumption **(H3)**, (3.13), (3.11) with $m = \gamma - 1$ and afterwards Remark 2.11, we obtain

$$\begin{aligned} &\|E(|u|^\gamma)u - E(|v|^\gamma)v\|_{L^{(\frac{\alpha+2}{\alpha+1},\infty)}} \\ &\leq \|E(|u|^\gamma)(u-v)\|_{L^{(\frac{\alpha+2}{\alpha+1},\infty)}} + \|E(|u|^\gamma - |v|^\gamma)v\|_{L^{(\frac{\alpha+2}{\alpha+1},\infty)}} \\ &\leq C \|E(|u|^\gamma)\|_{L^{(\frac{\alpha+2}{\alpha},\infty)}} \|u-v\|_{L^{(\alpha+2,\infty)}} + C \|E(|u|^\gamma - |v|^\gamma)\|_{L^{(l,\infty)}} \|v\|_{L^{(z,\infty)}} \\ &\leq C \| |u|^\gamma \|_{L^{(\frac{\alpha+2}{\alpha},\infty)}} \|u-v\|_{L^{(\alpha+2,\infty)}} + C \|(|u|^\gamma - |v|^\gamma)\|_{L^{(l,\infty)}} \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}} \\ &\leq C \| |u|^\gamma \|_{L^{(\frac{\alpha+2}{\alpha},\infty)}} \|u-v\|_{L^{(\alpha+2,\infty)}} \\ &\quad + C \| |u|^{\gamma-1} + |v|^{\gamma-1} \|_{L^{(\frac{z}{\gamma-1},\infty)}} \|u-v\|_{L^{(\alpha+2,\infty)}} \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}} \\ &\leq C \|u\|_{L^{(z,\infty)}}^\gamma \|u-v\|_{L^{(\alpha+2,\infty)}} \\ &\quad + C (\|u\|_{L^{(z,\infty)}}^{\gamma-1} + \|v\|_{L^{(z,\infty)}}^{\gamma-1}) \|u-v\|_{L^{(\alpha+2,\infty)}} \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}} \\ &\leq C \|\Lambda^s u\|_{L^{(\alpha+2,\infty)}}^\gamma \|u-v\|_{L^{(\alpha+2,\infty)}} \\ &\quad + C (\|\Lambda^s u\|_{L^{(\alpha+2,\infty)}}^{\gamma-1} + \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}}^{\gamma-1}) \|u-v\|_{L^{(\alpha+2,\infty)}} \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}} \\ &\leq C (\|\Lambda^s u\|_{L^{(\alpha+2,\infty)}}^\gamma + \|\Lambda^s v\|_{L^{(\alpha+2,\infty)}}^\gamma) \|u-v\|_{L^{(\alpha+2,\infty)}}. \end{aligned} \quad (3.14)$$

From the last inequality and Lemma 2.16, we obtain that

$$\begin{aligned}
J_2 &\leq |b|Ct^\beta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} (\|E(|u|^\gamma)u - E(|v|^\gamma)v\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)}) d\tau \\
&\leq |b|Ct^\beta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} (\|\Lambda^s u(\tau)\|_{L(\alpha+2, \infty)}^\gamma \\
&\quad + \|\Lambda^s v(\tau)\|_{L(\alpha+2, \infty)}^\gamma) \|u(\tau) - v(\tau)\|_{L(\alpha+2, \infty)} d\tau \\
&\leq |b|CM^\gamma d(u, v) t^{1-\frac{n}{d}\frac{\alpha}{\alpha+2}-\gamma\delta} \int_0^1 (1-s)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-(\beta+\gamma\delta)} ds \\
&\leq |b|C_2 M^\gamma d(u, v),
\end{aligned}$$

where in the last inequality we used that $\gamma\delta + \beta < 1$, $\frac{n\alpha}{d(\alpha+2)} < 1$, and

$$\frac{n}{d} \frac{\alpha}{\alpha+2} + \gamma\delta = 1. \quad (3.15)$$

For J_3 , we use Lemmas 2.16 and 2.12 to obtain

$$\begin{aligned}
J_3 &\leq |a|Ct^\delta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} \|\Lambda^s(|u(\tau)|^\alpha u(\tau))\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} d\tau \\
&\leq |a|Ct^\delta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} \|u(\tau)\|_{L(\alpha+2, \infty)}^\alpha \|\Lambda^s u(\tau)\|_{L(\alpha+2, \infty)} d\tau
\end{aligned}$$

In comparison with J_3 , the handling of J_4 requires some care due to the presence of the nonlocal operator E and the fractional derivative Λ^s . For that, we recall the parameters $z = \frac{\alpha+2}{\alpha}\gamma$ and $l = \frac{(\alpha+2)\gamma}{(\alpha+1)(\gamma-1)+1}$ and apply Leibniz's rule in the setting of weak- L^p (see [15, Theorem 6.1]) to estimate

$$\begin{aligned}
\|\Lambda^s(E(|u|^\gamma)u)\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} &\leq \|\Lambda^s(E(|u|^\gamma))\|_{L(l, \infty)} \|u\|_{L(z, \infty)} \\
&\quad + \|E(|u|^\gamma)\|_{L(\frac{\alpha+2}{\alpha}, \infty)} \|\Lambda^s u\|_{L(\alpha+2, \infty)} \quad (3.16)
\end{aligned}$$

Next, let $l_1 = \frac{z}{\gamma-1}$ and note that $\frac{1}{l} = \frac{1}{l_1} + \frac{1}{\alpha+2}$. Using **(H3)**, Lemma 2.13 and then Remark 2.11 in the R.H.S. of (3.16), we obtain that

$$\begin{aligned}
\|\Lambda^s(E(|u|^\gamma)u)\|_{L(\frac{\alpha+2}{\alpha+1}, \infty)} &\leq C\|\Lambda^s(|u|^\gamma)\|_{L(l, \infty)} \|u\|_{L(z, \infty)} \\
&\quad + C\| |u|^\gamma \|_{L(\frac{\alpha+2}{\alpha}, \infty)} \|\Lambda^s u\|_{L(\alpha+2, \infty)} \\
&\leq C\| |u|^{\gamma-1} \|_{L(l_1, \infty)} \|\Lambda^s u\|_{L(\alpha+2, \infty)} \|u\|_{L(z, \infty)} \\
&\quad + C\|u\|_{L(z, \infty)}^\gamma \|\Lambda^s u\|_{L(\alpha+2, \infty)} \\
&\leq C\|u\|_{L(z, \infty)}^{\gamma-1} \|\Lambda^s u\|_{L(\alpha+2, \infty)} \|u\|_{L(z, \infty)} \\
&\quad + C\|u\|_{L(z, \infty)}^\gamma \|\Lambda^s u\|_{L(\alpha+2, \infty)} \\
&\leq C\|\Lambda^s u\|_{L(\alpha+2, \infty)}^{\gamma+1} \quad (3.17)
\end{aligned}$$

Finally, Lemma 2.16 and estimate (3.17) yield

$$\begin{aligned} J_4 &\leq |b|Ct^\delta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} \|\Lambda^s(E(|u(\tau)|^\gamma)u(\tau))\|_{L^{\frac{\alpha+2}{\alpha+1},\infty}} d\tau \\ &\leq |b|Ct^\delta \int_0^t (t-\tau)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} \|\Lambda^s u(\tau)\|_{L^{\alpha+2,\infty}}^{\gamma+1} d\tau. \end{aligned}$$

Following the same ideas as before, using (3.12), (3.15), and the facts that $\beta\alpha + \delta < 1$, $\delta(\gamma + 1) < 1$, we have

$$J_3 + J_4 \leq |a|C_3M^{\alpha+1} + |b|C_4M^{\gamma+1}.$$

With the above estimates in hand we are able to prove existence of global solutions to (1.9). Indeed, suppose that $u_0 \in \mathcal{I}_\rho$. The four estimates above combined with assumptions (3.7) and (3.8) promptly allow us to show that Φ acts from X_M to X_M . In addition, the estimates for J_1 and J_2 together with (3.9) implies the existence of a positive constant $K_0 < 1$ such that

$$d(\Phi u, \Phi v) \leq K_0 d(u, v),$$

for any $u, v \in X_M$. The Banach fixed point theorem then gives us the desired result. $\square$

Remark 3.3. Our results also holds true for NLS-like equations with nonlinearity having two powers and double nonlocal operators E_1 and E_2 satisfying (H3). More precisely, in (1.1) we can consider the nonlinearity $aE_1(|u|^\alpha)u + bE_2(|u|^\gamma)u$. To do so, it is sufficient to estimate the terms J_1 and J_3 with the new term E_1 (see page 12) by following the arguments used to handle J_2 and J_4 .

Remark 3.4. Here we give a sufficient condition to initial data satisfying the assumptions in Theorem 3.2. Let $\varphi \in \dot{H}_{q,\infty}^\sigma \cap \dot{H}_{r,\infty}^\tau$, where

$$\sigma = -\frac{\beta d}{2} + \frac{n\alpha}{2(\alpha+2)}, \quad \frac{1}{q} = \frac{\beta d}{2n} + \frac{1}{2}, \quad \tau = s - \frac{\delta d}{2} + \frac{n\alpha}{2(\alpha+2)}, \quad \frac{1}{r} = \frac{\delta d}{2n} + \frac{1}{2},$$

with $\beta, \delta, \alpha, \gamma, s$ satisfying the assumptions in Theorem 3.2. Noting that

$$\sigma = n \left(\frac{1}{q'} - \frac{1}{\alpha+2} \right) \quad \text{and} \quad \tau - s = n \left(\frac{1}{r'} - \frac{1}{\alpha+2} \right),$$

and using Lemmas 2.10 and 2.16, we estimate

$$\begin{aligned} \|U(t)\varphi\|_{L^{\alpha+2,\infty}} &= \|\Lambda^{-\sigma}U(t)\Lambda^\sigma\varphi\|_{L^{\alpha+2,\infty}} \\ &\leq C\|U(t)\Lambda^\sigma\varphi\|_{L^{(q',\infty)}} \\ &\leq C|t|^{-\beta}\|\Lambda^\sigma\varphi\|_{L^{(q,\infty)}}. \end{aligned}$$

Analogously,

$$\begin{aligned} \|\Lambda^s U(t)\varphi\|_{L^{\alpha+2,\infty}} &= \|\Lambda^{s-\tau}U(t)\Lambda^\tau\varphi\|_{L^{\alpha+2,\infty}} \\ &\leq C\|U(t)\Lambda^\tau\varphi\|_{L^{(r',\infty)}} \\ &\leq C|t|^{-\delta}\|\Lambda^\tau\varphi\|_{L^{(r,\infty)}}. \end{aligned}$$

It follows that $U(t)\varphi \in X_\rho$ for some $\rho > 0$. In particular, the assumptions in Theorem 3.2 hold provided that $u_0 \in \dot{H}_{q,\infty}^\sigma \cap \dot{H}_{r,\infty}^\tau$ and ρ and M are small enough.

Remark 3.5. (Well-posedness) Under the assumptions of Theorem 3.2, if u_0 and v_0 are two tempered distributions in X_ρ and if u and v are the corresponding solutions then we easily see that there exists a constant $C > 0$ such that

$$d(u, v) \leq Cd(U(t)u_0, U(t)v_0),$$

which shows the continuous dependence of solutions with respect to initial data. Thus, we have in fact obtained a well-posedness result in the sense of Hadamard in our setting.

Remark 3.6. A few words of explanation concerning our assumptions in Theorem 3.2 are in order. For $r > 0$, let $\{r\}$ denotes the smallest integer bigger than or equal to r . Instead of assuming $0 < s < 1$, in [47] it was assumed

$$\{s\} < \alpha + 1. \quad (3.18)$$

In particular, when $s < 1$, (3.18) is equivalent to $\alpha > 0$, which brings no additional assumption on α . Hence our assumption is more restrictive than the one in [47]. On the other hand, the assumption $s < 1$ is equivalent to

$$\alpha > \frac{(n-2)\gamma}{\gamma+n}. \quad (3.19)$$

It is clear that if $n = 1$ or $n = 2$, (3.19) is always true. So, our assumption makes sense only in dimension $n \geq 3$ in which case we are indeed assuming that the relation between α and γ satisfies

$$\frac{(n-2)\gamma}{\gamma+n} < \alpha < \gamma.$$

In the sequel we will study some properties of the global solution obtained in Theorem 3.2. The first one concerns scattering in weak Lebesgue spaces.

Theorem 3.7. (Scattering) *Suppose that the assumptions in Theorem 3.2 hold and let u be the corresponding global solution with initial data u_0 . Then, there exists $u_+ \in \mathcal{I}_{\tilde{\rho}}$, for some $\tilde{\rho} > 0$, such that*

$$\|u(t) - U(t)u_+\|_{L(\alpha+2,\infty)} \leq Ct^{-\beta} \left(\|u\|_\beta^{\alpha+1} + \|u\|_{\delta,s}^\gamma \|u\|_\beta \right), \quad t > 0. \quad (3.20)$$

and

$$\|\Lambda^s(u(t) - U(t)u_+)\|_{L(\alpha+2,\infty)} \leq Ct^{-\delta} \left(\|u\|_\beta^\alpha \|u\|_{\delta,s} + \|u\|_{\delta,s}^{\gamma+1} \right), \quad t > 0. \quad (3.21)$$

In particular

$$\lim_{t \rightarrow \infty} \left(\|u(t) - U(t)u_+\|_{L(\alpha+2,\infty)} + \|\Lambda^s(u(t) - U(t)u_+)\|_{L(\alpha+2,\infty)} \right) = 0.$$

Proof. To simplify notation let us write $F(s) = a(|u|^\alpha u)(s) + b(E(|u|^\gamma)u)(s)$. From (1.9), we have, for $t > 1$,

$$U(-t)u(t) = u_0 + i \int_0^1 U(-s)F(s)ds + i \int_1^t U(-s)F(s)ds. \quad (3.22)$$

Let us show that the last integral on the right-hand side of (3.22) is convergent as $t \rightarrow \infty$. In fact, as in the proof of Theorem 3.2,

$$\begin{aligned} \int_1^t \|U(-s)F(s)\|_{L(\alpha+2,\infty)} ds &\leq C\|u\|_\beta^{\alpha+1} \int_1^t s^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-\beta(\alpha+1)} ds \\ &\quad + C\|u\|_{\delta,s}^\gamma \|u\|_\beta \int_1^t s^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-(\gamma\delta+\beta)} ds \\ &\leq CM^{\alpha+1} \int_1^t s^{-(1+\beta)} ds + CM^{\gamma+1} \int_1^t s^{-(1+\beta)} ds \\ &\leq C(M)(1-t^{-\beta}), \end{aligned}$$

where in the second inequality we have used (3.12) and (3.15). This implies that the distribution

$$u_+ := u_0 + i \int_0^\infty U(-s)F(s)ds$$

is well-defined. Note that

$$U(t)u_+ = U(t)u_0 + i \int_0^\infty U(t-s)F(s)ds.$$

We are going to show that $u_+ \in \mathcal{I}_{\tilde{\rho}}$, for some $\tilde{\rho} > 0$. First we claim that $\|u_+\|_\beta < \infty$. For that, it is sufficient to show that

$$\int_0^\infty \|U(t-s)F(s)\|_{L(\alpha+2,\infty)} ds \leq Ct^{-\beta}, \text{ for all } t > 0. \quad (3.23)$$

To establish (3.23) we split

$$\int_0^\infty U(t-s)F(s)ds = \int_0^t U(t-s)F(s)ds + \int_t^\infty U(t-s)F(s)ds. \quad (3.24)$$

The first integral on the right-hand side of (3.24) can be estimated as in the proof of Theorem 3.2. So that,

$$\begin{aligned} \int_0^t \|U(t-s)F(s)\|_{L(\alpha+2,\infty)} ds &\leq Ct^{-\beta} \left(\|u\|_\beta^{\alpha+1} + \|u\|_{\delta,s}^\gamma \|u\|_\beta \right) \\ &\leq C(M^{\alpha+1} + M^{\gamma+1})t^{-\beta}. \end{aligned} \quad (3.25)$$

For the second integral, by using (3.12) and (3.15), we deduce

$$\begin{aligned} \int_t^\infty \|U(t-s)F(s)\|_{L(\alpha+2,\infty)} ds &\leq Ct^{-\beta} M^{\alpha+1} \int_1^\infty (s-1)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-(\alpha\beta+\beta)} ds \\ &\quad + Ct^{-\beta} M^{\gamma+1} \int_1^\infty (s-1)^{-\frac{n}{d}\frac{\alpha}{\alpha+2}} s^{-(\gamma\delta+\beta)} ds \\ &= Ct^{-\beta} I_1 + Ct^{-\beta} I_2, \end{aligned} \quad (3.26)$$

where the integrals I_1 and I_2 are finite. Using (3.25) and (3.26), we obtain (3.23). From our calculations above, it is also clear that (3.20) holds.

By following the arguments above and using the estimates for J_3 and J_4 , it is not difficult to see that $\|u_+\|_{\delta,s} < \infty$ and that (3.21) also holds. The proof of the theorem is thus completed. $\square$

In next result, we investigate suitable conditions on the initial data so that solutions present decay faster than those in Theorem 3.2.

Theorem 3.8. *Under the hypotheses of Theorem 3.2.*

- (i) *Assume that u_0 satisfies $\|U(t)u_0\|_\mu < \infty$ for some $\mu \geq 0$ with $\alpha\beta + \mu < 1$. Assume also that there exist positive constants C_5 and C_6 such that*

$$|a|C_5M^\alpha + |b|C_6M^\gamma < 1, \quad (3.27)$$

where C_5 and C_6 are constants explicitly obtained in the calculations below. Then, the solution u given by Theorem 3.2 verifies the decay property $\|u\|_\mu < \infty$.

- (ii) *Assume that u_0 satisfies $\|U(t)u_0\|_{\nu,s} < \infty$ for some $\nu \geq 0$ with $\alpha\beta + \nu < 1$. Assume also that there exist positive constants C_7 and C_8 such that*

$$|a|C_7M^\alpha + |b|C_8M^\gamma < 1,$$

where C_7 and C_8 are constant explicitly obtained in the calculations below. Then, the solution u given by Theorem 3.2 satisfies the property $\|u\|_{\nu,s} < \infty$.

Proof. Working as in the estimates for J_1 and J_2 above, we can estimate

$$\begin{aligned} J_5 &:= |a|t^\mu \int_0^t \|U(t-\tau)(|u|^\alpha)(\tau)\|_{L(\alpha+2,\infty)} d\tau \\ &\leq |a|CM^\alpha t^\mu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\beta\alpha-\mu} d\tau \sup_{t>0} t^\mu \|u(t)\|_{L(\alpha+2,\infty)} \\ &\leq |a|C_5M^\alpha \|u\|_\mu, \end{aligned}$$

where in the last inequality we used (3.12) and the facts that $\frac{n\alpha}{d(\alpha+2)} < 1$ and $\beta\alpha + \mu < 1$.

Now using the same ideas,

$$\begin{aligned} J_6 &:= |b|t^\mu \int_0^t \|U(t-\tau)[E(|u|^\gamma)u(\tau)]\|_{L(\alpha+2,\infty)} d\tau \\ &\leq |b|CM^\gamma t^\mu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\delta\gamma-\mu} d\tau \sup_{t>0} t^\mu \|u(t)\|_{L(\alpha+2,\infty)} \\ &\leq |b|C_6M^\gamma \|u\|_\mu, \end{aligned}$$

where in the last inequality we used that $\frac{n\alpha}{d(\alpha+2)} < 1$, $\delta\gamma + \mu < 1$ and (3.15). Note that $\alpha\beta = \gamma\delta$, so the assumption $\alpha\beta + \mu < 1$ also gives $\gamma\delta + \mu < 1$.

We will reapply the contraction-mapping argument in order to get the desired property. Since $\|U(t)\varphi\|_\mu < \infty$, there exists $\sigma > 0$ such that $\|U(t)\varphi\|_\mu \leq$

σ . From hypothesis (3.27) we have that $1 - (|a|C_5M^\alpha + |b|C_6M^\gamma) > 0$. Hence, by choosing $K > 0$ such that

$$K \geq \frac{\sigma}{1 - (|a|C_5M^\alpha + |b|C_6M^\gamma)},$$

we deduce that

$$\sigma + |a|C_5M^\alpha K + |b|C_6M^\gamma K \leq K.$$

Now consider the following subspace $Y_{M,K} \subset X_M$:

$$Y_{M,K} = \{w \in X_M; \|w\|_\mu \leq K\}.$$

Observe that $(Y_{M,K}, d)$ is a nonempty complete metric space, with d as in Definition 3.1.

Let Φ be the integral operator defined in (3.10). Let us show that Φ maps $Y_{M,K}$ into itself and $\Phi : Y_{M,K} \rightarrow Y_{M,K}$ is a contraction. Suppose that $u \in Y_{M,K}$. Estimates for J_5 and J_6 yield

$$\|\Phi(u)\|_\mu \leq \sigma + |a|C_5M^\alpha K + |b|C_6M^\gamma K \leq K,$$

proving that $\Phi(Y_{M,K}) \subset Y_{M,K}$. Since the distance in $Y_{M,K}$ is that in X_M and we already proved that Φ is a contraction on X_M (see proof of Theorem 3.2) we have that Φ is also a contraction on $Y_{M,K}$. This implies that the solution u in X_M is also in $Y_{M,K}$, which means that $\|u\|_\mu < \infty$.

Now we turn to item (ii). Suppose that $\|U(t)\varphi\|_{\nu,s} \leq \sigma$ and choose $K > 0$ such that

$$\sigma + |a|C_7M^\alpha K + |b|C_8M^\gamma K \leq K.$$

By defining $Z_{M,K} \subset X_M$ as

$$Z_{M,K} = \{w \in X_M; \|w\|_{\nu,s} \leq K\},$$

we see that $(Z_{M,K}, d)$ is a nonempty complete metric space with the metric d as in Definition 3.1. Let us show that Φ maps $Z_{M,K}$ into itself and $\Phi : Z_{M,K} \rightarrow Z_{M,K}$ is a contraction. By assuming $u \in Z_{M,K}$ and slightly adapting the estimate for J_3 in Theorem 3.2, we easily arrive at

$$\begin{aligned} J_7 &:= |a|t^\nu \int_0^t \|\Lambda^s U(t-\tau)(|u(\tau)|^\alpha u(\tau))\|_{L^{(\alpha+2,\infty)}} d\tau \\ &\leq |a|Ct^\nu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \|u(\tau)\|_{L^{(\alpha+2,\infty)}}^\alpha \|\Lambda^s u(\tau)\|_{L^{(\alpha+2,\infty)}} d\tau \\ &\leq |a|Ct^\nu M^\alpha K \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\beta\alpha-\nu} d\tau \leq |a|C_7M^\alpha K, \end{aligned}$$

where we used (3.12), $\frac{n\alpha}{d(\alpha+2)} < 1$, and $\beta\alpha + \nu < 1$.

Now we proceed as in the estimate for J_4 to get

$$\begin{aligned} J_8 &:= |b|t^\nu \int_0^t \|\Lambda^s U(t-\tau)[E(|u(\tau)|^\gamma)u(\tau)]\|_{L(\alpha+2,\infty)} d\tau \\ &\leq |b|Ct^\nu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \|\Lambda^s u(\tau)\|_{L(\alpha+2,\infty)}^{\gamma+1} d\tau \\ &\leq |b|Ct^\nu M^\alpha K \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\delta\gamma-\nu} d\tau \leq |b|C_8 M^\gamma K, \end{aligned}$$

where we used (3.15), $\frac{n\alpha}{d(\alpha+2)} < 1$, and $\delta\gamma + \nu < 1$.

Using the estimates for J_7 and J_8 , it follows that

$$\|\Phi(u)\|_{\nu,s} \leq \sigma + |a|C_7 M^\alpha K + |b|C_8 M^\gamma K \leq K,$$

proving that $\Phi(Z_{M,K}) \subset Z_{M,K}$. The conclusion then follows as in the first part and we are done. $\square$

From another point of view, the solution u in Theorem 3.2 and Theorem 3.8 (i) satisfies $\|u(\cdot, t)\|_{L(\alpha+2,\infty)} = \mathcal{O}(t^{-\mu})$ provided that $\|U(t)u_0\|_{L(\alpha+2,\infty)} = \mathcal{O}(t^{-\mu})$ as $t \rightarrow \infty$, for $\mu = \beta$ and $\mu(\alpha+1) < 1$, respectively. In the sequel we provide a criterion for solutions to be asymptotically stable, which, in particular, assures that we can replace $\mathcal{O}(t^{-\mu})$ by $o(t^{-\mu})$ in the last two equalities.

Theorem 3.9. (Asymptotic stability) *Assume the hypotheses of Theorem 3.2 and that $u_0, v_0 \in \mathcal{I}_\rho$. For some $\mu \geq \beta$ with $\alpha\beta + \mu < 1$, suppose that $\|U(t)u_0\|_\mu < \infty$, $\|U(t)v_0\|_\mu < \infty$ and*

$$\lim_{t \rightarrow \infty} t^\mu \|U(t)(u_0 - v_0)\|_{L(\alpha+2,\infty)} = 0.$$

For $\mu > \beta$, assume that $|a|C_5 M^\alpha + |b|C_6 M^\gamma < 1$. For $\mu = \beta$, we have condition (3.9). Let u and v be the solutions of (1.9) with initial values u_0 and v_0 , respectively, given by Theorem 3.2. Then

$$\lim_{t \rightarrow \infty} t^\mu \|u(t) - v(t)\|_{L(\alpha+2,\infty)} = 0.$$

Proof. For $\mu \geq \beta$, we can estimate

$$\begin{aligned} t^\mu \|u(t) - v(t)\|_{L(\alpha+2,\infty)} &\leq t^\mu \|U(t)(u_0 - v_0)\|_{L(\alpha+2,\infty)} \\ &\quad + |a|CM^\alpha t^\mu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\beta\alpha-\mu} \tau^\mu \|u(\tau) \\ &\quad - v(\tau)\|_{L(\alpha+2,\infty)} d\tau \\ &\quad + |b|CM^\gamma t^\mu \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\delta\gamma-\mu} \tau^\mu \|u(\tau) \\ &\quad - v(\tau)\|_{L(\alpha+2,\infty)} d\tau \end{aligned}$$

Making the change of variables $\tau = tz$, we arrive at

$$\begin{aligned}
t^\mu \|u(t) - v(t)\|_{L(\alpha+2, \infty)} &\leq t^\mu \|U(t)(u_0 - v_0)\|_{L(\alpha+2, \infty)} \\
&\quad + |a|CM^\alpha \int_0^1 (1-z)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} z^{-\beta\alpha-\mu} (tz)^\mu \|u(tz) \\
&\quad - v(tz)\|_{L(\alpha+2, \infty)} dz \\
&\quad + |b|CM^\gamma \int_0^1 (1-z)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} z^{-\delta\gamma-\mu} (tz)^\mu \|u(tz) \\
&\quad - v(tz)\|_{L(\alpha+2, \infty)} dz
\end{aligned} \tag{3.28}$$

Denote

$$A = \limsup_{t \rightarrow \infty} t^\mu \|u(t) - v(t)\|_{L(\alpha+2, \infty)}.$$

For $\mu > \beta$ and $\mu = \beta$, we have that $A < \infty$ due to Theorems 3.8 and 3.2, respectively. Computing *limsup* in both sides of (3.28) and using Dominated Convergence Theorem, we obtain

$$\begin{aligned}
A &\leq 0 + |a|CM^\alpha \int_0^1 (1-z)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} z^{-\beta\alpha-\mu} dz A \\
&\quad + |b|CM^\gamma \int_0^1 (1-z)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} z^{-\delta\gamma-\mu} dz A \\
&= RA,
\end{aligned} \tag{3.29}$$

where $R = |a|C_1M^\alpha + |b|C_2M^\gamma < 1$ for $\mu = \beta$ and $R = |a|C_5M^\alpha + |b|C_6M^\gamma < 1$ for $\mu > \beta$. Thus, it follows that $A = 0$, which gives the desired conclusion. $\square$

Equation (1.1) has no scaling $u(x, t) \mapsto \lambda^m u(\lambda x, \lambda^2 t)$, for any $m \in \mathbb{R}$. This fact prevents the existence of self-similar solutions to (1.1). Alternatively, we will prove that (1.1) admits a class of asymptotically self-similar solutions with respect to the scaling of (1.2). Resembling results in the L^p -setting can be found in [47].

For that matter, first note that Theorem 3.2 with $b = 0$ gives global mild solutions in weak- L^p spaces for (1.2) (see also [7, 43]), i.e., solutions v of the integral equation

$$v(t) = U(t)v_0 + i \int_0^t U(t-s)(a|v|^\alpha v)(s)ds, \tag{3.30}$$

satisfying $\|v\|_\beta \leq M$. Moreover, with a slight modification in the proof, we only need to assume $\|U(t)v_0\|_\beta \leq \rho$ instead of $v_0 \in \mathcal{I}_\rho$.

In the next theorem, we compare the mild solutions of (1.1) and (1.2). In fact, we are going to prove that solutions of (1.2) attract those of (1.1) as $t \rightarrow \infty$, depending on a suitable condition for the difference of the initial values $\psi = u_0 - v_0$.

Theorem 3.10. *Under the hypotheses of Theorem 3.2. Let u be the corresponding solution of (1.9) with initial value $u_0 \in \mathcal{I}_\rho$ given by Theorem 3.2. Let v be the solution of (3.30) (i.e., (1.9) with $b = 0$) with initial value v_0 , such that $\|U(t)v_0\|_\beta \leq \rho$, also given by Theorem 3.2. Suppose further that u_0 satisfies the*

hypotheses of part (ii) of Theorem 3.8 with some $\nu > \delta$ such that $\gamma\nu + \beta < 1$. Then, we have that

$$\lim_{t \rightarrow \infty} t^\beta \|u(t) - v(t)\|_{L(\alpha+2, \infty)} = 0, \quad (3.31)$$

provided that

$$\lim_{t \rightarrow \infty} t^\beta \|U(t)(u_0 - v_0)\|_{L(\alpha+2, \infty)} = 0. \quad (3.32)$$

Proof. First note that

$$\begin{aligned} t^\beta \|u(t) - v(t)\|_{L(\alpha+2, \infty)} &\leq t^\beta \|U(t)(u_0 - v_0)\|_{L(\alpha+2, \infty)} \\ &\quad + |a| t^\beta \left\| \int_0^t U(t-\tau)(|u|^\alpha u - |v|^\alpha v)(\tau) d\tau \right\|_{L(\alpha+2, \infty)} \\ &\quad + |b| t^\beta \left\| \int_0^t U(t-\tau)[E(|u|^\gamma u)](\tau) d\tau \right\|_{L(\alpha+2, \infty)} \end{aligned}$$

Proceeding as in the estimates for J_1 and J_2 above, we obtain

$$\begin{aligned} t^\beta \|u(t) - v(t)\|_{L(\alpha+2, \infty)} &\leq t^\beta \|U(t)(u_0 - v_0)\|_{L(\alpha+2, \infty)} \\ &\quad + |a| C M^\alpha t^\beta \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\beta\alpha-\beta} \tau^\beta \|u(\tau) - v(\tau)\|_{L(\alpha+2, \infty)} d\tau \\ &\quad + |b| C M^\gamma t^\beta \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \|\Lambda^s u(\tau)\|_{L(\alpha+2, \infty)}^\gamma \|u(\tau)\|_{L(\alpha+2, \infty)} d\tau \\ &:= A_1(t) + A_2(t) + A_3(t). \end{aligned} \quad (3.33)$$

From Theorem 3.8 part (ii), we have that $\|u\|_{\nu, s} = \sup_{\tau > 0} \tau^\nu \|\Lambda^s u(\tau)\|_{L(\alpha+2, \infty)} < \infty$ and then

$$\begin{aligned} A_3(t) &\leq C |b| t^\beta \int_0^t (t-\tau)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\gamma\nu-\beta} d\tau \|u\|_{\nu, s}^\gamma \|u\|_\beta \\ &= t^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})+1-\gamma\delta-\gamma(\nu-\delta)} C |b| \int_0^1 (1-z)^{-\frac{n}{d}(\frac{\alpha}{\alpha+2})} \tau^{-\gamma\nu-\beta} d\tau \|u\|_{\nu, s}^\gamma \|u\|_\beta \\ &= C t^{-\gamma(\nu-\delta)} \|u\|_{\nu, s}^\gamma \|u\|_\beta \rightarrow 0, \text{ as } t \rightarrow \infty. \end{aligned} \quad (3.34)$$

Taking $\mathcal{H} = \limsup_{t \rightarrow \infty} t^\beta \|u(t) - v(t)\|_{L(\alpha+2, \infty)}$, working in the same spirit of the proof of Theorem 3.9 and using (3.33)–(3.34), we obtain

$$\mathcal{H} \leq 0 + |a| C_1 M^\alpha \mathcal{H} + 0,$$

which gives (3.31), because $0 \leq \mathcal{H} < \infty$ and $|a| C_1 M^\alpha < 1$. $\square$

We finish by showing the existence of mild solutions of (1.1) that are asymptotically self-similar at infinite, with respect to the scaling of the single-power Schrödinger equation (1.2).

Corollary 3.11. (*Asymptotic self-similarity*) In addition to the hypotheses of Theorem 3.10, assume that v_0 is a homogeneous distribution of degree $-d/\alpha$ and that $u_0 = v_0 + \omega$ with ω satisfying (3.32), e.g., $\omega \in L^{(\frac{\alpha+2}{\alpha+1}, \infty)}$. Let v be the self-similar solution of (3.30) corresponding to the initial value v_0 . Then, the solution u of (1.9) with initial value u_0 satisfies

$$\lim_{t \rightarrow \infty} t^\beta \|u(t) - v(t)\|_{L^{(\alpha+2, \infty)}} = 0. \quad (3.35)$$

In other words, one obtains a class of solutions of (1.9) that are attracted in the sense of (3.35) to the self-similar solution v of (3.30).

Proof. For $\omega \in L^{(\frac{\alpha+2}{\alpha+1}, \infty)}$, using Lemma 2.16 and noting that $\beta < \frac{n\alpha}{d(\alpha+2)}$, we have that

$$0 \leq t^\beta \|U(t)\omega\|_{L^{(\alpha+2, \infty)}} \leq Ct^{\beta - \frac{n\alpha}{d(\alpha+2)}} \|\omega\|_{L^{(\frac{\alpha+2}{\alpha+1}, \infty)}} \rightarrow 0, \text{ as } t \rightarrow \infty, \quad (3.36)$$

and then $\omega = u_0 - v_0$ satisfies (3.32). If this condition is verified, then (3.35) follows from Theorem 3.10. Furthermore, notice that v is self-similar because v_0 is homogeneous of degree $-d/\alpha$ (see, for instance, [7, 9, 43]). The proof of the corollary is thus completed. $\square$

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