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INFINITE-ENERGY SOLUTIONS FOR SCHRÖDINGER-TYPE EQUATIONS WITH A NONLOCAL TERM

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Abstract. We study the Cauchy problem associated with nonlinear Schrödinger-type equations with a nonlocal term in $\mathbb{R}^n$. Existence and uniqueness of local and global solutions are established in spaces which allow singular initial data. Scattering, asymptotic stability, and decay rates are also proved.

1. Introduction

This paper is concerned with the initial-value problem (IVP) associated with Schrödinger-type equations of the form

$$\begin{cases} i\partial_t u + Lu = \chi|u|^\rho u + bE(|u|^\rho)u, \\ u(x,0) = u_0(x), \end{cases} \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function, χ and b are real constants, L and E are linear operators, and ρ is a positive real number.

Our main goal is to give sufficient conditions on the operators L and E that allow us to establish the local and global well-posedness of the IVP (1.1) for a "sufficiently large" class of initial data that includes homogeneous functions. Self-similar solutions will be obtained as a consequence.

The usual example where (1.1) appears is in the case $b = 0$ and $L = \Delta$, where Δ stands for the Laplacian operator. In such a situation, (1.1) reduces

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to the well-known nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \chi |u|^\rho u, \tag{1.2}$$

which has a central role in the theory of nonlinear dispersive equations and emerges in many fields in applied physics (see for example [19], [31], [41], and [44]). Another example that reduces to (1.1) is the n-dimensional ($n \geq 2$) Davey–Stewartson (DS) system

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi |u|^\rho u + bu\partial_{x_1}\varphi, \\ \partial_{x_1}^2 \varphi + m\partial_{x_2}^2 \varphi + \sum_{j=3}^n \partial_{x_j}^2 \varphi = \partial_{x_1}(|u|^\rho), \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $u = u(x,t)$ is a complex-valued function and $\varphi = \varphi(x,t)$ is a real-valued function. The parameters χ, b, and δ are real constants and m is a positive number. The Davey–Stewartson system was derived in [14] in the context of water waves. Since then, many works concerning various different topics including solvability of the initial- and initial-boundary-value problems, blow-up solutions, existence of periodic solutions, stability of standing waves, etc., are available in the current literature (we refer the reader to [13], [17], [20], [21], [22], [23], [24], [25], [33], [35], [36], [29], and references therein). As is well-known, considering $n = 2$ and $\rho = 2$, the Davey–Stewartson system generalizes the one-dimensional cubic Schrödinger equation

$$i\partial_t u + \partial_x^2 u = \chi |u|^2 u$$

in water-waves modelling.

On one hand, under suitable conditions on L and E, the Cauchy problem (1.1) has attracted the attention of many researchers in the framework of the Sobolev spaces $H^s(\mathbb{R}^n)$, which are the appropriate spaces to study (1.1) (see for instance [9], [10], [11], [26], [27], and references therein), and it has been shown to be well-understood by now. On the other hand, it seems to be quite natural and interesting to look for solutions of (1.1) which are invariant by scaling. Since in this situation the initial data is required to be a homogeneous function, the local and global theories in the Sobolev spaces fail in providing such solutions. As a consequence, it is necessary to use suitable functional spaces which allow homogeneous Cauchy data. This approach was initiated in [7] and [12] for the Schrödinger equation.

Although our results are inspired on the Davey–Stewartson system, our analysis goes beyond. Indeed, throughout the paper, we assume that L is a pseudo-differential operator defined via its Fourier transform by

$$\widehat{Lu}(\xi) = q(\xi)\widehat{u}(\xi), \tag{1.4}$$

where we have the following:

(H1) The function q is real and homogeneous of degree d; that is,

$$q(\lambda\xi) = \lambda^d q(\xi), \qquad \lambda > 0.$$

(H2) The function $G(x) = \int_{\mathbb{R}^n} e^{i(x\xi+q(\xi))}\, d\xi$ belongs to $L^\infty(\mathbb{R}^n)$.

The assumption on the operator E is that

(H3) E is bounded from $L^{(p,\infty)}(\mathbb{R}^n)$ to itself, for all p satisfying $1 < p < \infty$.

Here and throughout the paper, $L^{(p,\infty)}(\mathbb{R}^n)$ stands for the weak Lebesgue space (see [4] or Section 2 for a brief review). In most examples, the operator E is shown to be bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Hence, by using the real interpolation method, one obtains the assumption (H3) (see Section 4).

To study the IVP (1.1) we use its equivalent integral formulation

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)(\chi|u|^\rho u + buE(|u|^\rho))(s)\, ds, \tag{1.5}$$

where $U(t)u_0$ is the solution of the linear problem

$$\begin{cases} i\partial_t u + Lu = 0, \\ u(x,0) = u_0(x), \end{cases} \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}; \tag{1.6}$$

that is,

$$U(t)u_0(x) = \int_{\mathbb{R}^n} e^{i(x\xi+tq(\xi))}\widehat{u}_0(\xi)\, d\xi. \tag{1.7}$$

Remark 1.1. From Stone's theorem, the operator $U(t)$ defines a unitary group on $H^s(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. In particular,

$$\|U(t)u_0\|_{L^2} = \|u_0\|_{L^2}. \tag{1.8}$$

The plan of the paper is the following. In Section 2, we introduce some notation and give two preliminary lemmas. The first one is concerned with the boundedness of the unitary group $U(t)$ in $L^{(p,\infty)}(\mathbb{R}^n)$. This result is crucial in our analysis below. The second lemma regards the boundedness of the integral part of (1.5) in our functional spaces, which is necessary to apply the Banach fixed-point theorem. In Section 3, we prove our main results. In particular, we establish local and global existence, asymptotic stability, decay, and existence of self-similar solutions. Finally, in Section 4, we give applications of our results to some physical models. In particular, we consider the Schrödinger equation, the Davey–Stewartson system, the Grimshaw system, and the Shrira system.

2. Notation and preliminaries

Let us begin this section by introducing the notation used throughout the paper. We use C to denote various constants that may vary line by line. We denote by $\|\cdot\|_{L^p}$, $1 \le p \le \infty$, the usual Lebesgue L^p-norm. The Fourier transform of a function $f = f(x)$ is defined by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

The inverse Fourier transform of a function $g = g(\xi)$ is denoted by $(\mathcal{F}^{-1}g)(x) = \check{g}(x)$. In $\mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions) the Fourier transform is understood in the usual sense. $\mathcal{S}(\mathbb{R}^n)$ will denote the class of all Schwartz functions.

The weak Lebesgue spaces $L^{(p,\infty)} = L^{(p,\infty)}(\mathbb{R}^n)$, $1 \le p < \infty$, are defined as

$$L^{(p,\infty)} = \{f : \mathbb{R}^n \to \mathbb{C} \text{ measurable} : \|f\|_{L^{(p,\infty)}} := \sup_{\lambda>0} \lambda\alpha(\lambda, f)^{1/p} < \infty\},$$

where

$$\alpha(\lambda, f) = \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}), \quad \text{and} \quad \mu \text{ is the Lebesgue measure.}$$

As is well-known, there exists an equivalent norm in $L^{(p,\infty)}$, $1 < p < \infty$, such that $L^{(p,\infty)}$ becomes a Banach space. Moreover,

$$L^p \hookrightarrow L^{(p,\infty)}$$

with continuous embedding.

We observe that if $1 < p, q, r < \infty$, then the Hölder inequality

$$\|fg\|_{L^{(r,\infty)}} \le \|f\|_{L^{(p,\infty)}} \|g\|_{L^{(q,\infty)}}, \qquad \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r},$$

holds (see [32]). Also, if $1 \le r < \infty$ and $1 < q, p < \infty$, then the Young inequality

$$\|f * g\|_{L^{(q,\infty)}} \le C\|g\|_{L^r} \|f\|_{L^{(p,\infty)}}, \qquad \tfrac{1}{q} = \tfrac{1}{p} + \tfrac{1}{r} - 1, \tag{2.1}$$

is valid (see [16, page 21]).

Next we prove a Bernstein-type inequality.

Proposition 2.1 (Bernstein's inequality). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that* $\operatorname{supp} \widehat{f} \subset B(0,R)$*, where $B(0,R)$ denotes the ball in $\mathbb{R}^n$ with radius R centered at the origin. If $1 < p \le q < \infty$, then there exists $C > 0$ such that*

$$\|f\|_{L^{(q,\infty)}} \le C R^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^{(p,\infty)}}.$$

Proof. The proof is similar to that for Lebesgue spaces. For the sake of completeness, we bring it here. We claim that it suffices to assume $R = 1$. Indeed, suppose we have proved the proposition for $R = 1$; then if $\operatorname{supp} \widehat{f} \subset B(0,R)$, define $g(x) = f(R^{-1}x)$. We have $\widehat{g}(\xi) = R^n \widehat{f}(R\xi)$ and $\operatorname{supp} \widehat{g} \subset B(0,1)$. Since

$$\|g\|_{L^{(r,\infty)}} = \|f(R^{-1}\cdot)\|_{L^{(r,\infty)}} = R^{\frac{n}{r}} \|f\|_{L^{(r,\infty)}}, \qquad 1 < r < \infty,$$

the conclusion then follows just by applying the result for g.

Assume then that $\operatorname{supp} \widehat{f} \subset B(0,1)$. Take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\widehat{\varphi} \equiv 1$ on $B(0,1)$ and $\widehat{\varphi} \equiv 0$ on $\mathbb{R}^n \setminus B(0,2)$. Since $\widehat{f} = \widehat{\varphi}\widehat{f}$, we have $f = \varphi * f$. In view of the Young inequality (2.1), we deduce

$$\|f\|_{L^{(q,\infty)}} \le C \|\varphi\|_{L^r} \|f\|_{L^{(p,\infty)}}, \qquad \tfrac{1}{q} = \tfrac{1}{p} + \tfrac{1}{r} - 1.$$

Because $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we obtain $\varphi \in L^r$, $1 < r < \infty$, and thus the result follows. $\square$

Let $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^n)$ be a function satisfying $0 \le \widehat{\varphi} \le 1$, $\widehat{\varphi} = 1$ if $|\xi| \le 1$, and $\widehat{\varphi} = 0$ if $|\xi| > 2$. Define

$$\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi), \qquad \widehat{\psi}_j(\xi) = \widehat{\psi}(2^{-j}\xi), \quad j \in \mathbb{Z},$$

so that

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(\xi) = 1, \ \ \xi \ne 0, \quad \text{and} \quad \operatorname{supp}(\widehat{\psi}_j) \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}.$$

Next, we define the Littlewood–Paley multiplier Δ_j as

$$\Delta_j f = (\widehat{\psi}_j \widehat{f})^\vee = \psi_j * f, \qquad j \in \mathbb{Z}. \tag{2.2}$$

Also, let $\widehat{\eta}$ be another smooth function supported in $\{1/4 < |\xi| < 4\}$ such that $\widehat{\eta} = 1$ on $\operatorname{supp}(\widehat{\psi})$. We define $\widetilde{\Delta}_j$ like Δ_j with η instead of ψ. Thus, the identity

$$\widetilde{\Delta}_j \Delta_j = \Delta_j \tag{2.3}$$

holds.

We define the *weak* homogeneous Besov space to be

$$\dot{B}^{s,q}_{p,\infty} = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}^{s,q}_{p,\infty}} = \Big(\sum_{j\in\mathbb{Z}} 2^{2js} \|\Delta_j f\|^q_{L^{(p,\infty)}} \Big)^{1/q} < \infty \Big\},$$

where $s \in \mathbb{R}$, and $1 \le p, q < \infty$. In particular,

$$\|f\|_{\dot{B}^{0,1}_{p,\infty}} = \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^{(p,\infty)}}.$$

It is not difficult to see that $\dot{B}^{0,1}_{p,\infty}$ is continuously imbedded in $L^{(p,\infty)}$ (see, e.g., [4, Theorem 6.3.1]); that is, there exists $C > 0$ such that

$$\|f\|_{L^{(p,\infty)}} \le C\|f\|_{\dot{B}^{0,1}_{p,\infty}}, \qquad f \in \dot{B}^{0,1}_{p,\infty}. \tag{2.4}$$

From now on, we denote

$$\alpha := \frac{d}{\rho} - \frac{n}{\rho+2} \quad \text{and} \quad \beta := \frac{d}{\rho} - \frac{n(\rho+1)}{\rho+2}. \tag{2.5}$$

Definition 2.2. *Let $0 < \rho < \infty$ and $0 < T < \infty$. We denote by E_α and $E^T_{\alpha,\beta}$ the Banach spaces*

$$E_\alpha = \{u : |t|^{\alpha/d} u \in L^\infty(\mathbb{R}; L^{(\rho+2,\infty)})\}, \tag{2.6}$$

$$E^T_{\alpha,\beta} = \{u : |t|^{(\alpha-\beta)/d} u \in L^\infty((-T,T); L^{(\rho+2,\infty)})\}, \tag{2.7}$$

with respective norms

$$\|u\|_\alpha = \sup_{-\infty<t<+\infty} |t|^{\alpha/d} \|u(t)\|_{L^{(\rho+2,\infty)}}, \tag{2.8}$$

and

$$\|u\|_{\alpha,\beta,T} = \sup_{-T<t<T} |t|^{(\alpha-\beta)/d} \|u(t)\|_{L^{(\rho+2,\infty)}}. \tag{2.9}$$

The next lemma establishes the boundedness of the linear group $U(t)$ in weak Lebesgue spaces.

Lemma 2.3. *Let $1 < p < 2$. If p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$, then there exists a constant $C = C(n,p) > 0$ such that*

$$\|U(t)\phi\|_{L^{(p',\infty)}} \le C|t|^{-\frac{n}{d}(\frac{2}{p}-1)} \|\phi\|_{L^{(p,\infty)}}, \tag{2.10}$$

for all $\phi \in L^{(p,\infty)}(\mathbb{R}^n)$ and all $t \neq 0$.

Proof. Without loss of generality assume $t > 0$. From (1.8) it follows that $U(t)$ is bounded from $L^2(\mathbb{R}^n)$ into itself. Moreover, since

$$U(t)\phi(x) = t^{-n/d} G(t^{-1/d}(\cdot)) * \phi(x),$$

it follows from assumption (H2) and the Young inequality that

$$\|U(t)\phi\|_{L^\infty} \le t^{-n/d}\|\phi\|_{L^1}.$$

The Riesz–Thorin interpolation theorem establishes (2.10) if we replace the weak Lebesgue spaces by the usual Lebesgue spaces (this was already established in [18]). Then the real interpolation method gives us the desired result (see e.g. [30, Theorem 3.1] and [37]). See also [5] for a similar result. $\square$

Lemma 2.4. *Let $1 < \rho < \infty$ and let B be defined as*

$$B(u) = i\int_0^t U(t-s)(\chi|u|^\rho u + buE(|u|^\rho))(s)\, ds. \tag{2.11}$$

(i) *If $\frac{n\rho}{d} < \frac{\rho+2}{\rho+1}$, then there exists a positive constant $K_{\alpha,\beta}$ such that*

$$\|B(u) - B(v)\|_{\alpha,\beta,T} \le K_{\alpha,\beta} T^\gamma \Big(\|u\|^\rho_{\alpha,\beta,T} + \|v\|^\rho_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T}, \tag{2.12}$$

for all $u, v \in E^T_{\alpha,\beta}$, where $\gamma = 1 - \frac{(\alpha-\beta)(\rho+1)}{d}$.

(ii) *If $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d} < \rho + 2$, then there exists a positive constant K_α such that*

$$\|B(u) - B(v)\|_\alpha \le K_\alpha(\|u\|^\rho_\alpha + \|v\|^\rho_\alpha)\|u-v\|_\alpha, \tag{2.13}$$

for all $u, v \in E_\alpha$.

Proof. Without loss of generality, we assume $t > 0$. We first prove inequality (2.12). By Lemma 2.3, we have

$$\begin{aligned}
\|B(u) - B(v)\|_{L^{(\rho+2,\infty)}} &\le |\chi|\int_0^t \|U(t-s)(|u|^\rho u - |v|^\rho v)(s)\|_{L^{(\rho+2,\infty)}}\, ds \\
&\quad + |b|\int_0^t \|U(t-s)(uE(|u|^\rho) - vE(|v|^\rho))(s)\|_{L^{(\rho+2,\infty)}}\, ds \\
&\le |\chi|\int_0^t (t-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}\|(|u|^\rho u - |v|^\rho v)(s)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}\, ds \\
&\quad + |b|\int_0^t (t-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}\|(uE(|u|^\rho) - vE(|v|^\rho))(s)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}\, ds \\
&= I + II.
\end{aligned} \tag{2.14}$$

We now estimate I. By using the well-known inequality

$$||u|^\rho u - |v|^\rho v| \leq C(|u|^\rho + |v|^\rho)|u - v|,$$

and then Holder's inequality, we obtain

$$\begin{aligned} &\||u|^\rho u - |v|^\rho v\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \leq C\Big\|\Big(|u|^\rho + |v|^\rho\Big)|u-v|\Big\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\ &\leq C\Big(\||u|^\rho\|_{L^{(\frac{\rho+2}{\rho},\infty)}} + \||v|^\rho\|_{L^{(\frac{\rho+2}{\rho},\infty)}}\Big)\|u-v\|_{L^{(\rho+2,\infty)}} \\ &\leq C\Big(\|u\|^\rho_{L^{(\rho+2,\infty)}} + \|v\|^\rho_{L^{(\rho+2,\infty)}}\Big)\|u-v\|_{L^{(\rho+2,\infty)}}. \end{aligned} \tag{2.15}$$

Thus, for $0 < t < T$,

$$\begin{aligned} I &\leq C\int_0^t (t-s)^{-\frac{n}{d}(\frac{2(\rho+1)}{\rho+2}-1)} \\ &\qquad \times \Big(\|u(s)\|^\rho_{L^{(\rho+2,\infty)}} + \|v(s)\|^\rho_{L^{(\rho+2,\infty)}}\Big)\|(u-v)(s)\|_{L^{(\rho+2,\infty)}}\,ds \\ &\leq C\Big(\|u\|^\rho_{\alpha,\beta,T} + \|v\|^\rho_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T} \\ &\qquad \times \int_0^t (t-s)^{-\frac{n}{d}(\frac{2(\rho+1)}{\rho+2}-1)} s^{-\frac{\alpha-\beta}{d}(\rho+1)}\,ds \\ &\leq C\Big(\|u\|^\rho_{\alpha,\beta,T} + \|v\|^\rho_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T} t^{-\frac{\alpha-\beta}{d}} t^\gamma \\ &\qquad \times \int_0^1 (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha-\beta}{d}(\rho+1)}\,ds, \end{aligned} \tag{2.16}$$

where in the last inequality we have used that

$$\tfrac{\alpha-\beta}{d} = \tfrac{n}{d}\Big(\tfrac{2(\rho+1)}{\rho+2} - 1\Big).$$

Since $\frac{n\rho}{d} < \frac{\rho+2}{\rho+1}$, we easily obtain $\frac{\alpha-\beta}{d} < 1$ and $\frac{\alpha-\beta}{d}(\rho+1) < 1$. Hence, the integral in (2.16) is finite, and thus

$$I \leq K_{\alpha,\beta} T^\gamma \Big(\|u\|^\rho_{\alpha,\beta,T} + \|v\|^\rho_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T} t^{-\frac{\alpha-\beta}{d}}. \tag{2.17}$$

To estimate II, we observe that by writing

$$E(|u|^\rho)u - E(|v|^\rho)v = E(|u|^\rho)(u-v) + E(|u|^\rho - |v|^\rho)v,$$

using Hölder's inequality and hypothesis (H3), we get

$$\begin{aligned} &\|E(|u|^\rho)u - E(|v|^\rho)v\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\ &\leq \|E(|u|^\rho)\|_{L^{(\frac{\rho+2}{\rho},\infty)}}\|u-v\|_{L^{(\rho+2,\infty)}} + \|E(|u|^\rho - |v|^\rho)\|_{L^{(\frac{\rho+2}{\rho},\infty)}}\|v\|_{L^{(\rho+2,\infty)}} \end{aligned}$$

$$\leq C\Big(\|u\|^{\rho}_{L^{(\rho+2,\infty)}}\|u-v\|_{L^{(\rho+2,\infty)}} + \||u|^{\rho}-|v|^{\rho}\|_{L^{(\frac{\rho+2}{\rho},\infty)}}\|v\|_{L^{(\rho+2,\infty)}}\Big).$$

Next, since

$$||u|^{\rho}-|v|^{\rho}| \leq C(|u|^{\rho-1}+|v|^{\rho-1})|u-v|,$$

we have

$$\begin{aligned}
&\|E(|u|^{\rho})u - E(|v|^{\rho})v\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\
&\quad\leq C\Big(\|u\|^{\rho}_{L^{(\rho+2,\infty)}}\|u-v\|_{L^{(\rho+2,\infty)}} + \||u|^{\rho-1}+|v|^{\rho-1}\|_{L^{(\frac{\rho+2}{\rho-1},\infty)}} \\
&\qquad\qquad \times \|u-v\|_{L^{(\rho+2,\infty)}}\|v\|_{L^{(\rho+2,\infty)}}\Big) \\
&\quad\leq C\Big(\|u\|^{\rho}_{L^{(\rho+2,\infty)}}\|u-v\|_{L^{(\rho+2,\infty)}} + \Big(\|u\|^{\rho-1}_{L^{(\rho+2,\infty)}} + \|v\|^{\rho-1}_{L^{(\rho+2,\infty)}}\Big) \\
&\qquad\qquad \times \|u-v\|_{L^{(\rho+2,\infty)}}\|v\|_{L^{(\rho+2,\infty)}}\Big) \\
&\quad\leq C\Big(\|u\|^{\rho}_{L^{(\rho+2,\infty)}} + \|v\|^{\rho}_{L^{(\rho+2,\infty)}}\Big)\|u-v\|_{L^{(\rho+2,\infty)}}. \qquad (2.18)
\end{aligned}$$

Therefore, the estimate for II follows exactly in the same way as that for I; that is,

$$II \leq K_{\alpha,\beta}T^{\gamma}\Big(\|u\|^{\rho}_{\alpha,\beta,T} + \|v\|^{\rho}_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T}t^{-\frac{\alpha-\beta}{d}}. \qquad (2.19)$$

Gathering together (2.14), (2.17), and (2.19), we obtain, for $0 < t < T$,

$$t^{\frac{\alpha-\beta}{d}}\|B(u)-B(v)\|_{L^{(\rho+2,\infty)}} \leq K_{\alpha,\beta}T^{\gamma}\Big(\|u\|^{\rho}_{\alpha,\beta,T} + \|v\|^{\rho}_{\alpha,\beta,T}\Big)\|u-v\|_{\alpha,\beta,T},$$

and the proof of part (i) is completed.

To prove (2.13), we proceed as in (2.12). Indeed, from (2.15) and (2.18), we have

$$\begin{aligned}
\|B(u)-B(v)\|_{L^{(\rho+2,\infty)}} &\leq C\int_0^t (t-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)} \\
&\qquad \times \Big(\|u(s)\|^{\rho}_{L^{(\rho+2,\infty)}} + \|v(s)\|^{\rho}_{L^{(\rho+2,\infty)}}\Big)\|(u-v)(s)\|_{L^{(\rho+2,\infty)}}\,ds \\
&\leq C(\|u\|^{\rho}_{\alpha}+\|v\|^{\rho}_{\alpha})\|u-v\|_{\alpha}\int_0^t (t-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds \\
&\leq C(\|u\|^{\rho}_{\alpha}+\|v\|^{\rho}_{\alpha})\|u-v\|_{\alpha}t^{-\frac{n}{d}(\frac{2(\rho+1)}{\rho+2}-1)}t^{-\frac{\alpha}{d}(\rho+1)}t \\
&\qquad \times \int_0^1 (1-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds.
\end{aligned}$$

From the definition of α, we see that

$$\frac{n}{d}\Big(\frac{2(\rho+1)}{\rho+2}-1\Big)+\frac{\alpha}{d}\rho=1. \tag{2.20}$$

Moreover, since $\frac{\rho+2}{\rho+1}<\frac{n\rho}{d}$, we obtain $\frac{\alpha}{d}(\rho+1)<1$, and from $\frac{n\rho}{d}<\rho+2$, we have $\frac{n}{d}(\frac{2(\rho+1)}{\rho+2}-1)<1$. So,

$$\begin{aligned}
&\|B(u)-B(v)\|_{L^{(\rho+2,\infty)}}\\
&\le C(\|u\|_\alpha^\rho+\|v\|_\alpha^\rho)\|u-v\|_\alpha t^{-\frac{\alpha}{d}}\int_0^1(1-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds\\
&=K_\alpha(\|u\|_\alpha^\rho+\|v\|_\alpha^\rho)\|u-v\|_\alpha t^{-\frac{\alpha}{d}}.
\end{aligned}$$

This completes the proof of the lemma. □

Remark 2.5. The restriction $\rho>1$ has appeared only when we estimate the term II in (2.14). Hence, in the case $b=0$, the assumption (H3) is not necessary and Lemma 2.4 holds for $0<\rho<\infty$.

3. Main results

In this section we prove our main results. We start with a local well-posedness result. For the nonlinear Schrödinger equation (1.2), a similar approach is used in [5]. Our result extends that one. More precisely, we have the following.

Theorem 3.1 (Local Existence)**.** *Let $1<\rho<\infty$ and $\frac{n\rho}{d}<\frac{\rho+2}{\rho+1}$. If $\phi\in L^{(\frac{\rho+2}{\rho+1},\infty)}$, then there exist $0<T<\infty$ and a unique solution $u\in E^T_{\alpha,\beta}$ of the integral equation* (1.5) *with $T=T(\|\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}})$. Moreover, if $\phi_n\in L^{(\frac{\rho+1}{\rho+2},\infty)}$ is a sequence of functions satisfying $\phi_n\to\phi$ in $L^{(\frac{\rho+1}{\rho+2},\infty)}$, then there exist $0<T_0<\infty$ and $n_0\in\mathbb{N}$ such that, for $n\ge n_0$, the solutions u_n and u of the integral equation* (1.5) *with respective initial data ϕ_n and ϕ lie in $E^{T_0}_{\alpha,\beta}$, and $u_n\to u$, as $n\to\infty$, in $E^{T_0}_{\alpha,\beta}$. In addition, the flow map $\phi\mapsto u$ from $L^{(\frac{\rho+2}{\rho+1},\infty)}$ to $E^{T_0}_{\alpha,\beta}$, is Lipschitz continuous.*

Proof. The proof is based on the Banach fixed-point theorem. We consider the integral operator

$$(\Phi u)(t)=U(t)\phi+(Bu)(t), \tag{3.1}$$

where B is defined as in (2.11).

Let $\overline{B}(0,2R)$ be the closed ball in $E^T_{\alpha,\beta}$ with radius $2R$ and centered at the origin. We will show that there exist $R>0$ and $T>0$ such that Φ maps $\overline{B}(0,2R)\subset E^T_{\alpha,\beta}$ into itself and $\Phi:\overline{B}(0,2R)\to\overline{B}(0,2R)$ is a contraction.

First, we take $u\in\overline{B}(0,2R)$ and prove that $\|\Phi u\|_{\alpha,\beta,T}\le 2R$. From (3.1) we conclude that

$$\|\Phi u\|_{\alpha,\beta,T}\le\|U(\cdot)\phi\|_{\alpha,\beta,T}+\|Bu\|_{\alpha,\beta,T}. \tag{3.2}$$

Since $\frac{\alpha-\beta}{d}-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)=0$, in view of Lemma 2.3, we have

$$\|U(\cdot)\phi\|_{\alpha,\beta,T}\le\|\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}. \tag{3.3}$$

Now, from Lemma 2.4 and the fact that $u\in\overline{B}(0,2R)$, we get

$$\|Bu\|_{\alpha,\beta,T}\le K_{\alpha,\beta}T^{\gamma}\|u\|^{\rho+1}_{\alpha,\beta,T}\le K_{\alpha,\beta}T^{\gamma}2^{\rho+1}R^{\rho+1}, \tag{3.4}$$

where $\gamma=1-\frac{\alpha-\beta}{d}(\rho+1)>0$. From inequalities (3.2)–(3.4), we obtain

$$\|\Phi u\|_{\alpha,\beta,T}\le\|\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}+K_{\alpha,\beta}T^{\gamma}2^{\rho+1}R^{\rho+1}. \tag{3.5}$$

By taking $R=\|\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}$ and then $T>0$ such that

$$K_{\alpha,\beta}T^{\gamma}2^{\rho+1}R^{\rho}\le\frac{1}{3^{\rho}}, \tag{3.6}$$

we have

$$\|\Phi u\|_{\alpha,\beta,T}\le R+\frac{R}{3^{\rho}}\le 2R.$$

Using the same arguments as above and the definitions of T and R in (3.6), one proves that Φ contracts in $\overline{B}(0,2R)$. An application of the Banach fixed-point theorem then gives the existence of a unique solution of (1.5) in $\overline{B}(0,2R)$.

To prove the continuous dependence, we take ϕ and ϕ_n in $L^{(\frac{\rho+2}{\rho+1},\infty)}$ such that $\phi_n\to\phi$ in $L^{(\frac{\rho+2}{\rho+1},\infty)}$. For each n, there exist $T_n>0$ and a unique $u_n\in E^{T_n}_{\alpha,\beta}$ satisfying

$$\|u_n\|_{\alpha,\beta,T_n}\le 2\|\phi_n\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \tag{3.7}$$

and

$$u_n(t)=U(t)\phi_n+(Bu_n)(t). \tag{3.8}$$

Also there exists a unique $u\in E^{T_0}_{\alpha,\beta}$ satisfying

$$\|u\|_{\alpha,\beta,T_0}\le 2R \tag{3.9}$$

and

$$u(t) = U(t)\phi + (Bu)(t), \tag{3.10}$$

where R and T_0 are defined as in (3.6) with T_0 instead of T. From (3.6), since T_n varies continuously with $\|\phi_n\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}$, without loss of generality, we may assume $T_n \geq T_0$ for n large enough (if necessary we can take a smaller T_0). Thus, both u_n and u are defined in the interval $[0, T_0]$ and belong to $E^{T_0}_{\alpha,\beta}$.

Next, we shall prove that $\|u_n - u\|_{\alpha,\beta,T_0} \to 0$, as $n \to \infty$. From identities (3.8) and (3.10), we obtain

$$\|u_n - u\|_{\alpha,\beta,T_0} \leq \|U(\cdot)(\phi_n - \phi)\|_{\alpha,\beta,T_0} + \|Bu_n - Bu\|_{\alpha,\beta,T_0}. \tag{3.11}$$

By applying Lemmas 2.3 and 2.4,

$$\begin{aligned} \|u_n - u\|_{\alpha,\beta,T_0} \leq \|\phi_n - \phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} &+ K_{\alpha,\beta}T_0^{\gamma}\|u_n - u\|_{\alpha,\beta,T_0} \\ &\times \Big(\|u_n\|^{\rho}_{\alpha,\beta,T_0} + \|u\|^{\rho}_{\alpha,\beta,T_0}\Big). \end{aligned} \tag{3.12}$$

Now it suffices to prove that

$$K_{\alpha,\beta}T_0^{\gamma}\Big(\|u_n\|^{\rho}_{\alpha,\beta,T_0} + \|u\|^{\rho}_{\alpha,\beta,T_0}\Big) \leq C < 1. \tag{3.13}$$

In fact, from inequalities (3.12) and (3.13) it follows that

$$\|u_n - u\|_{\alpha,\beta,T_0} \leq \frac{1}{(1-C)}\|\phi_n - \phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}, \tag{3.14}$$

which yields the result. In order to prove (3.13) we use inequalities (3.7) and (3.9) to obtain

$$\begin{aligned} K_{\alpha,\beta}T_0^{\gamma}(\|u_n\|^{\rho}_{\alpha,\beta,T_0} &+ \|u\|^{\rho}_{\alpha,\beta,T_0}) \\ &\leq K_{\alpha,\beta}T_0^{\gamma}\Big(2^{\rho}\|\phi_n\|^{\rho}_{L^{(\frac{\rho+2}{\rho+1},\infty)}} + 2^{\rho}\|\phi\|^{\rho}_{L^{(\frac{\rho+2}{\rho+1},\infty)}}\Big) \qquad (3.15) \\ &= K_{\alpha,\beta}T_0^{\gamma}2^{\rho}\Big(\|\phi_n\|^{\rho}_{L^{(\frac{\rho+2}{\rho+1},\infty)}} + R^{\rho}\Big). \qquad (3.16) \end{aligned}$$

Since $\|\phi_n - \phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \to 0$ as $n \to \infty$, we can choose n_0 large enough such that for $n > n_0$, we have $\|\phi_n\|_{L^{(\frac{\rho+2}{\rho+1})}} < \varepsilon + R$ with $0 < \varepsilon < R$. Thus, $\|\phi_n\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} < 2R$. By using the definition of T_0 in (3.6), we deduce

$$K_{\alpha,\beta}T_0^{\gamma}(\|u_n\|^{\rho}_{\alpha,\beta,T_0} + \|u\|^{\rho}_{\alpha,\beta,T_0}) \leq K_{\alpha,\beta}T_0^{\gamma}2^{\rho}R^{\rho}(2^{\rho}+1)$$

$$= \frac{2^\rho + 1}{3^\rho 2} < \frac{2^\rho}{3^\rho} = \left(\frac{2}{3}\right)^\rho < 1. \qquad (3.17)$$

The last part of the theorem follows from inequality (3.14).

□

Remark 3.2. The condition $\frac{n\rho}{d} < \frac{\rho+2}{\rho+1}$ implies that $\rho < \frac{-(n-d)+\sqrt{(n-d)^2+8nd}}{2n}$. Thus, in addition to $1 < \rho < \infty$, we must have

$$1 < \rho < \frac{-(n-d) + \sqrt{(n-d)^2 + 8nd}}{2n}.$$

As a consequence, the existence of ρ satisfying the conditions of Theorem 3.1 is restricted to the case $n < \frac{3}{2}d$. In particular, if $d = 2$, we must have $n < 3$. Of course, if $b = 0$, such a restriction is not needed.

Next, we turn to the question of global existence. Unfortunately, the results are proved under a "smallness condition," and the global existence for arbitrary large initial data remains an open problem.

Theorem 3.3 (Global Existence)**.** *Let* $1 < \rho < \infty$ *and* $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d} < \rho + 2$. *Assume that* ϕ *is a distribution satisfying* $\|U(t)\phi\|_\alpha \leq \epsilon$, *where* $\epsilon > 0$ *is sufficiently small. Then,*

(i) *The integral equation* (1.5) *has a unique solution* $u \in E_\alpha$ *satisfying* $\|u\|_\alpha \leq 2\epsilon$.

(ii) *If* ϕ_n *is a sequence of distributions such that* $\|U(t)\phi_n - U(t)\phi\|_\alpha \to 0$, *as* $n \to \infty$, *and* u_n *and* u *are the solutions of the integral equation* (1.5) *with respective initial data* ϕ_n *and* ϕ, *then* $u_n \to u$ *in* E_α.

Proof. To prove (i), we consider the integral operator

$$(\Phi u)(t) = U(t)\phi + (Bu)(t),$$

with B defined in (2.11). As in Theorem 3.1, we use the Banach fixed-point theorem to find a function $u \in \overline{B}(0, 2\epsilon) \subset E_\alpha$ satisfying $\Phi u = u$, for some $\epsilon > 0$ small enough.

For any $u \in \overline{B}(0, 2\epsilon)$, using the hypothesis that $\|U(t)\phi\|_\alpha \leq \epsilon$ and Lemma 2.4, we conclude that

$$\|\Phi u\|_\alpha \leq \epsilon + \|Bu\|_\alpha \leq \epsilon + K_\alpha 2^{\rho+1}\epsilon^{\rho+1}.$$

By choosing $\epsilon > 0$ such that $K_\alpha 2^{\rho+1}\epsilon^\rho < 1$, we promptly see that $\|\Phi u\|_\alpha \leq 2\epsilon$. This proves that $\Phi : \overline{B}(0, 2\epsilon) \to \overline{B}(0, 2\epsilon)$ is well defined. A similar analysis also shows that Φ is a contraction. The Banach fixed-point theorem then gives the existence result. The rest of the proof follows as in Theorem 3.1, so we omit the details. □

Next we prove the existence of self-similar solutions. To make clear what it means, we observe that if $u(x,t)$ is a solution of the equation given in (1.1), so is $u_\lambda(x,t) = \lambda^{d/\rho}u(\lambda x, \lambda^d t)$, for any $\lambda > 0$. A *self-similar* solution is a solution of (1.1) such that

$$u(x,t) = u_\lambda(x,t), \ \forall \ \lambda > 0. \tag{3.18}$$

If (3.18) holds, then

$$u(x,0) = u_\lambda(x,0) = \lambda^{d/\rho}u(\lambda x, 0), \ \forall \ \lambda > 0.$$

This means that $u(x,0)$ must be homogeneous of degree $-d/\rho$.

Suppose that ϕ is homogeneous of degree $-d/\rho$; that is,

$$\phi(\lambda x) = \lambda^{-\frac{d}{\rho}}\phi(x), \ \forall \ \lambda > 0. \tag{3.19}$$

It is easy to see that

$$U(t)\phi(x) = \lambda^{\frac{d}{\rho}}U(\lambda^d t)\phi(\lambda x), \ \forall \ \lambda > 0.$$

Hence, by taking $\lambda = t^{-\frac{1}{d}}$, we obtain

$$U(t)\phi(x) = t^{-\frac{1}{\rho}}U(1)\phi(t^{-\frac{1}{d}}x)$$

and

$$t^{\frac{\alpha}{d}}\|U(t)\phi\|_{L^{(\rho+2,\infty)}} = t^{\frac{\alpha}{d}-\frac{1}{\rho}+\frac{n}{d(\rho+2)}}\|U(1)\phi\|_{L^{(\rho+2,\infty)}} = \|U(1)\phi\|_{L^{(\rho+2,\infty)}}, \tag{3.20}$$

where we have used (2.5).

Theorem 3.4. *Let $1 < \rho < \infty$ and $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d} < \rho + 2$. Assume that ϕ is homogeneous of degree $-d/\rho$ and $\|U(1)\phi\|_{L^{(\rho+2,\infty)}} \leq \epsilon$, where $\epsilon > 0$ is sufficiently small. Then the solution u obtained in Theorem* 3.3 *is self-similar.*

Proof. The proof immediately follows from Theorem 3.3, taking into account (3.20). □

We now look for some reasonable condition on ϕ to get $\|U(1)\phi\|_{L^{(\rho+2,\infty)}} < \infty$. Here, we use the ideas in [39] and [42].

Theorem 3.5. *Let $1 < \rho < \infty$ and $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d} < \rho + 2$. Assume that ϕ is homogeneous of degree $-d/\rho$ and $\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} < \infty$, with Δ_0 defined as in* (2.2). *Then,*

$$\|U(1)\phi\|_{L^{(\rho+2,\infty)}} \leq C\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} < \infty.$$

In particular, from (3.20), $\|U(t)\phi\|_\alpha < \infty$.

Proof. In view of (2.4), it suffices to prove that there exists a constant $C > 0$ satisfying

$$\|U(1)\phi\|_{\dot{B}^{0,1}_{\rho+2,\infty}} \le C\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}. \tag{3.21}$$

Let us write $U(1)\phi = F_1 + F_2$, where

$$F_1 = U(1)(\varphi * \phi), \qquad F_2 = U(1)((1-\widehat{\varphi})^\vee * \phi),$$

with φ given in Section 2. We first estimate F_2. Since $U(1)$ commutes with Δ_j, we see from Lemma 2.3 that

$$\begin{aligned}\|F_2\|_{\dot{B}^{0,1}_{\rho+2,\infty}} &= \sum_{j\in\mathbb{Z}} \|\Delta_j U(1)((1-\widehat{\varphi})^\vee * \phi)\|_{L^{(\rho+2,\infty)}} \\ &= \sum_{j\in\mathbb{Z}} \|U(1)\Delta_j((1-\widehat{\varphi})^\vee * \phi)\|_{L^{(\rho+2,\infty)}} \\ &\le \sum_{j\in\mathbb{Z}} \|\Delta_j((1-\widehat{\varphi})^\vee * \phi)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}.\end{aligned}$$

From the definition of Δ_j, we observe that $\Delta_j((1-\widehat{\varphi})^\vee * \phi) = 0$ for $j \le -1$. Thus, using (2.3), the identity $\widetilde{\Delta}_j\Delta_j((1-\widehat{\varphi})^\vee * \phi) = \widetilde{\Delta}_j((1-\widehat{\varphi})^\vee) * \Delta_j\phi$, and Young's inequality, we have

$$\begin{aligned}\|F_2\|_{\dot{B}^{0,1}_{\rho+2,\infty}} &\le \sum_{j=0}^{\infty} \|\Delta_j((1-\widehat{\varphi})^\vee * \phi)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\ &= \sum_{j=0}^{\infty} \|\widetilde{\Delta}_j\Delta_j((1-\widehat{\varphi})^\vee * \phi)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\ &= \sum_{j=0}^{\infty} \|\widetilde{\Delta}_j((1-\widehat{\varphi})^\vee) * \Delta_j\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \\ &\le \sum_{j=0}^{\infty} \|\widetilde{\Delta}_j((1-\widehat{\varphi})^\vee)\|_{L^1}\|\Delta_j\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}.\end{aligned} \tag{3.22}$$

But since $\widetilde{\Delta}_j((1-\widehat{\varphi})^\vee) = \eta_j - \eta_j * \varphi$, and $\eta_j(x) = 2^{jn}\eta(2^j x)$, we have

$$\begin{aligned}\|\widetilde{\Delta}_j((1-\widehat{\varphi})^\vee)\|_{L^1} &\le \|\eta_j\|_{L^1} + \|\eta_j * \varphi\|_{L^1} \\ &\le (1+\|\varphi\|_{L^1})\|\eta_j\|_{L^1} = (1+\|\varphi\|_{L^1})\|\eta\|_{L^1} = C.\end{aligned} \tag{3.23}$$

Moreover, using the homogeneity of ϕ, we deduce that

$$\Delta_j\phi(x) = (\psi_j * \phi)(x) = 2^{j\frac{d}{\rho}}(\psi * \phi)(2^j x) = 2^{j\frac{d}{\rho}}\Delta_0\phi(2^j x).$$

Hence,

$$\|\Delta_j\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} = 2^{j\frac{d}{\rho}}\|\Delta_0\phi(2^j\cdot)\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} = 2^{j\left(\frac{d}{\rho}-n\frac{\rho+1}{\rho+2}\right)}\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}. \tag{3.24}$$

From (3.22)–(3.24), we then get

$$\|F_2\|_{\dot{B}^{0,1}_{\rho+2,\infty}} \le C\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \sum_{j=0}^{\infty} 2^{j\left(\frac{d}{\rho}-n\frac{\rho+1}{\rho+2}\right)}.$$

Thanks to the inequality $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d}$, the above sum is finite.

Next, we estimate F_1. By taking into account that $\Delta_j\varphi = 0$, $j \ge 2$, we have

$$\begin{aligned}
\|F_1\|_{\dot{B}^{0,1}_{\rho+2,\infty}} &= \sum_{j\in\mathbb{Z}} \|\Delta_j U(1)(\varphi * \phi)\|_{L^{(\rho+2,\infty)}} \le \sum_{j\le 1} \|U(1)\Delta_j(\varphi*\phi)\|_{L^{(\rho+2,\infty)}} \\
&= \sum_{j\le 1} \|(U(1)\varphi) * \Delta_j\phi\|_{L^{(\rho+2,\infty)}} \le \sum_{j\le 1} \|U(1)\varphi\|_{L^1}\|\Delta_j\phi\|_{L^{(\rho+2,\infty)}} \\
&\le C\|\Delta_0\phi\|_{L^{(\rho+2,\infty)}} \sum_{j\le 1} 2^{j\left(\frac{d}{\rho}-n\frac{1}{\rho+2}\right)},
\end{aligned}$$

the sum being finite now due to the inequality $\frac{n\rho}{d} < \rho+2$. Because $\frac{\rho+2}{\rho+1} < \rho+2$, an application of the Bernstein inequality (see Proposition 2.1) then gives the desired estimate, that is,

$$\|F_1\|_{\dot{B}^{0,1}_{\rho,\infty}} \le C\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}}.$$

This completes the proof of the theorem. □

In the next corollary, $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$.

Corollary 3.6. *Let* $1 < \rho < \infty$ *and* $\frac{\rho+2}{\rho+1} < \frac{n\rho}{d} < \rho+2$. *Let* $\Phi \in C^k(\mathbb{S}^{n-1})$ *and suppose that*

$$\frac{\rho+2}{\rho+1}\Big(\frac{d}{\rho}+k\Big) > n. \tag{3.25}$$

Define the homogeneous function

$$\phi(x) = \epsilon\Phi\Big(\frac{x}{|x|}\Big)|x|^{-d/\rho},$$

where $\epsilon > 0$ *is sufficiently small. Then, the global solution* u *given in Theorem* 3.3 *with initial data* ϕ *is self-similar.*

Proof. From Theorems 3.4 and 3.5 it suffices to show that, for $\epsilon = 1$, $\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} < \infty$. To prove this, it is enough to establish that

$$\|\Delta_0\phi\|_{L^{(\frac{\rho+2}{\rho+1},\infty)}} \leq C\|\Phi\|_{C^k},$$

for some positive constant C. The continuous embedding of $L^{\frac{\rho+2}{\rho+1}}$ in $L^{(\frac{\rho+2}{\rho+1},\infty)}$ then implies that we have to show the estimate

$$\|\Delta_0\phi\|_{L^{\frac{\rho+2}{\rho+1}}} \leq C\|\Phi\|_{C^k}. \tag{3.26}$$

But since $\frac{d}{\rho} < n\frac{\rho+1}{\rho+2} < n$, estimate (3.26) follows as in the proof of Lemma 1 in [39] (see also [38]). $\square$

Remark 3.7. If $k = n$, then (3.25) is obviously satisfied.

Our next result is concerned with nonlinear scattering.

Theorem 3.8 (Scattering)**.** *Let u be the global solution of the integral equation* (1.5) *given by Theorem* 3.3 *corresponding to the initial data ϕ. Then, there exists $u_\pm \in L^{(\rho+2,\infty)}$ satisfying*

$$\|U(t)u_\pm\|_\alpha < \infty, \tag{3.27}$$

and

$$\|u(t) - U(t)u_\pm\|_{L^{(\rho+2,\infty)}} \leq C|t|^{-\frac{\alpha}{d}}\|u\|_\alpha^{\rho+1}, \quad t \neq 0. \tag{3.28}$$

In particular,

$$\lim_{t\to\pm\infty} \|u(t) - U(t)u_\pm\|_{L^{(\rho+2,\infty)}} = 0.$$

In addition, for any $\delta > 0$,

$$\lim_{t\to\pm\infty} |t|^{\frac{\alpha}{d}-\delta}\|u(t) - U(t)u_\pm\|_{L^{(\rho+2,\infty)}} = 0.$$

Proof. We only prove the existence of u_+. For short, let us denote $F(s) = (\chi|u|^\rho u + buE(|u|^\rho))(s)$. From (1.5), we have, for $t > 1$,

$$U(-t)u(t) = \phi + i\int_0^1 U(-s)F(s)\,ds + i\int_1^t U(-s)F(s)\,ds. \tag{3.29}$$

As in the proof of Lemma 2.4-(ii),

$$\begin{aligned}\int_1^t \|U(-s)F(s)\|_{L^{(\rho+2,\infty)}}\,ds &\leq C\|u\|_\alpha^{\rho+1}\int_1^t s^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds\\ &\leq C(2\epsilon)^{\rho+1}\int_1^t s^{-\left(1+\frac{\alpha}{d}\right)}\,ds \leq C(\epsilon)(1 - t^{\frac{-\alpha}{d}}),\end{aligned}$$

where in the second inequality we have used (2.20). This implies that there exists $u_+ \in L^{(\rho+2,\infty)}$ such that the left-hand side of (3.29) converges, in $L^{(\rho+2,\infty)}$, to u_+, as $t \to \infty$; that is,

$$u_+ = \phi + i\int_0^\infty U(-s)F(s)\,ds.$$

Since

$$U(t)u_+ = U(t)\phi + i\int_0^\infty U(t-s)F(s)\,ds,$$

to prove (3.27) we only have to show that the integral part belongs to E_α. We can write

$$\int_0^\infty U(t-s)F(s)\,ds = \int_0^t U(t-s)F(s)\,ds + \int_t^\infty U(t-s)F(s)\,ds. \quad (3.30)$$

The first integral on the right-hand side of (3.30) can be estimated as in the proof of Lemma 2.4-(ii), so that

$$\int_0^t \|U(t-s)F(s)\|_{L^{(\rho+2,\infty)}}\,ds \le Ct^{-\frac{\alpha}{d}}\|u\|_\alpha^{\rho+1}. \quad (3.31)$$

For the second integral, we have

$$\begin{aligned}\int_t^\infty &\|U(t-s)F(s)\|_{L^{(\rho+2,\infty)}}\,ds \\ &\le Ct^{-\frac{\alpha}{d}}\|u\|_\alpha^{\rho+1}\int_1^\infty (s-1)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds. \quad (3.32)\end{aligned}$$

Thus, it remains to prove that the above integral is finite. Fix any $r_0 > 1$ and write

$$\begin{aligned}&\int_1^\infty (s-1)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds \\ &= \int_1^{r_0} (s-1)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds + \int_{r_0}^\infty (s-1)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\frac{\alpha}{d}(\rho+1)}\,ds \\ &= I_1 + I_2.\end{aligned}$$

Since $\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right) < 1$ (see (2.20)), it follows that I_1 is finite. Now, for I_2,

$$I_2 = \int_{r_0}^\infty \left(\frac{s}{s-1}\right)^{\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)}s^{-\left(\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)+\frac{\alpha\rho}{d}+\frac{\alpha}{d}\right)}\,ds \le C\int_{r_0}^\infty s^{-\left(1+\frac{\alpha}{d}\right)}\,ds,$$

where we have used that the function $s \mapsto (\frac{s}{s-1})^{\frac{n}{d}(\frac{2(\rho+1)}{\rho+2}-1)}$ is bounded. This last integral is obviously finite. Estimate (3.28) is proved similarly. The proof of the theorem is completed. $\square$

Next, we shall prove two results concerning the behavior of the solutions given in Theorems 3.3 and 3.1. These results are similar to those established in [5] for the Schrödinger equation.

Theorem 3.9 (Asymptotic Stability)**.** *Let $\delta > 0$ satisfying $\delta + \frac{\alpha}{d}(\rho+1) < 1$. Let u and v be two global solutions of the integral equation* (1.5) *given by Theorem* 3.3, *corresponding to the initial conditions ϕ and $\widetilde{\phi}$, respectively. If*

$$\lim_{|t|\to\infty} |t|^{\frac{\alpha}{d}+\delta} \|U(t)(\phi - \widetilde{\phi})\|_{L^{(\rho+2,\infty)}} = 0,$$

then

$$\lim_{|t|\to\infty} |t|^{\frac{\alpha}{d}+\delta} \|u(t) - v(t)\|_{L^{(\rho+2,\infty)}} = 0.$$

Proof. Assume $t > 0$. Note that, from (2.15) and (2.18),

$$\begin{aligned}\|u(t) - v(t)\|_{L^{(\rho+2,\infty)}} &\leq \|U(t)(\phi - \widetilde{\phi})\|_{L^{(\rho+2,\infty)}} \\ &\quad + \int_0^t (t-s)^{-\frac{n}{d}\left(\frac{2(\rho+1)}{\rho+2}-1\right)} \Big(\|u(s)\|^{\rho}_{L^{(\rho+2,\infty)}} + \|v(s)\|^{\rho}_{L^{(\rho+2,\infty)}} \Big) \\ &\qquad \times \|u(s) - v(s)\|_{L^{(\rho+2,\infty)}}\, ds.\end{aligned}$$

Since $\|u\|_{\alpha}, \|v\|_{\alpha} \leq 2\epsilon$, we deduce

$$\begin{aligned}\|u(t) - v(t)\|_{L^{(\rho+2,\infty)}} &\leq \|U(t)(\phi - \widetilde{\phi})\|_{L^{(\rho+2,\infty)}} \\ &\quad + C2^{\rho+1}\epsilon^{\rho} t^{-\frac{\alpha}{d}-\delta} \int_0^1 \Big\{ (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha}{d}(\rho+1)-\delta} (st)^{\frac{\alpha}{d}+\delta} \\ &\qquad \times \|u(st) - v(st)\|_{L^{(\rho+2,\infty)}} \Big\}\, ds, \end{aligned} \tag{3.33}$$

where we have used that

$$\frac{n}{d}\Big(\frac{2(\rho+1)}{\rho+2} - 1\Big) = \frac{\alpha - \beta}{d} = 1 - \frac{\alpha}{d}\rho = \frac{n\rho}{d(\rho+2)}.$$

Now we define

$$\Lambda = \limsup_{t\to\infty} t^{\frac{\alpha}{d}+\delta} \|u(t) - v(t)\|_{L^{(\rho+2,\infty)}} \geq 0.$$

It is enough to prove that $\Lambda = 0$. Assume for the sake of contradiction that $\Lambda > 0$. From (3.33) we see that

$$\Lambda \leq \Big(C2^{\rho+1}\epsilon^{\rho}\int_0^1 (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha}{d}(\rho+1)-\delta}\, ds\Big)\Lambda.$$

From the above inequality we then obtain

$$C2^{\rho+1}\epsilon^{\rho}\int_0^1 (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha}{d}(\rho+1)-\delta}\, ds \geq 1.$$

This is a contradiction, since $\epsilon > 0$ can be chosen to be small enough. Hence $\Lambda = 0$ and the theorem is proved. $\square$

Theorem 3.10 (Decay)**.** *Let* $\gamma = 1 - \frac{(\alpha-\beta)(\rho+1)}{d} > 0$ *and* $\delta > -\gamma$*. Let* u *and* v *be two local solutions of the integral equation* (1.5) *given by Theorem* 3.1*, corresponding to the initial conditions* ϕ *and* $\widetilde{\phi}$*, respectively. If*

$$\lim_{|t|\to 0} |t|^{\frac{\alpha-\beta}{d}-\delta}\|U(t)(\phi-\widetilde{\phi})\|_{L^{(\rho+2,\infty)}} = 0,$$

then

$$\lim_{|t|\to 0} |t|^{\frac{\alpha-\beta}{d}-\delta}\|u(t)-v(t)\|_{L^{(\rho+2,\infty)}} = 0.$$

Proof. Assume $t > 0$. The proof is similar to that of Theorem 3.9. Indeed, by noting that

$$\gamma = \frac{\alpha-\beta}{d} - \delta - \frac{\alpha-\beta}{d} - \frac{\alpha-\beta}{d}(\rho+1) + \delta + 1,$$

we may write

$$\begin{aligned} t^{\frac{\alpha-\beta}{d}-\delta}\|u(t)-v(t)\|_{L^{(\rho+2,\infty)}} &\leq t^{\frac{\alpha-\beta}{d}-\delta}\|U(t)(\phi-\widetilde{\phi})\|_{L^{(\rho+2,\infty)}} \\ &+ C2^{\rho+1}\epsilon^{\rho}t^{\gamma}\int_0^1 (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha-\beta}{d}(\rho+1)+\delta}(st)^{\frac{\alpha-\beta}{d}-\delta}\|u(st)-v(st)\|_{L^{(\rho+2,\infty)}}\, ds. \end{aligned} \tag{3.34}$$

By writing

$$\Lambda = \limsup_{t\to 0} t^{\frac{\alpha-\beta}{d}-\delta}\|u(t)-v(t)\|_{L^{(\rho+2,\infty)}} < \infty,$$

we get, after taking the limit in (3.34), as $t \to 0$,

$$0 \leq \Lambda \leq \Big(\Lambda C2^{\rho+1}\epsilon^{\rho}\int_0^1 (1-s)^{-\frac{\alpha-\beta}{d}} s^{-\frac{\alpha-\beta}{d}(\rho+1)+\delta}\, ds\Big)\lim_{t\to 0} t^{\gamma} = 0,$$

because $\Lambda < \infty$. This completes the proof. $\square$

Remark 3.11. According to Remark 2.5, in the case $b = 0$, the range $0 < \rho < \infty$ is allowed, so that all of our results in this section are valid, a priori, for $0 < \rho < \infty$.

4. Applications

In this section we apply our results to some well-known equations.

4.1. **The Nonlinear Schrödinger Equation.** We consider the nonlinear Schrödinger (NLS) equation

$$i\partial_t u + Lu = \chi|u|^\rho u, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \tag{4.1}$$

where χ is a real constant, $\rho > 0$, and L is an operator satisfying assumptions (H1) and (H2). Since $b = 0$, the assumption (H3) is not needed in this situation.

In the case $L = \Delta$, where Δ stands for the Laplacian operator, (4.1) reads as the standard NLS equation. In the same spirit of our work, the NLS equation was studied in [5], [7], [8], and [6]. In this case, all of our results in Section 3 apply (recall that $0 < \rho \leq 1$ can also be included).

Another particular case of interest is the so-called nonelliptic NLS equation, in which

$$Lu = \sum_{j=1}^{n} a_j \frac{\partial^2 u}{\partial x_j^2}, \tag{4.2}$$

and at least two of the a_j have opposite sign. Here, we have $d = 2$. To fix ideas let us take $n = 2$ and $q(\xi) = \xi_1^2 - \xi_2^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Existence of solutions for (4.1), with L as in (4.2), in the classical Sobolev spaces, was addressed in [17], [18], and quite recently in [34]. In particular, for $\rho = 2$, by using the Fourier restriction method, the author in [34] has obtained sharp bilinear estimates in the Bourgain spaces $X_{s,b}$ for $s \geq 0$. Once again, all of our results in Section 3 apply in this case.

4.2. **The Davey–Stewartson system.** Let us consider the Davey–Stewartson (DS) system

$$\begin{cases} iu_t + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi|u|^\rho u + bu\partial_{x_1}\varphi, \\ \partial_{x_1}^2\varphi + m\partial_{x_2}^2\varphi + \sum_{j=3}^n \partial_{x_j}^2\varphi = \partial_{x_1}(|u|^\rho), \end{cases} \tag{4.3}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n)$, $n \geq 2$, $u = u(x,t)$ is a complex-valued function, $\varphi = \varphi(x,t)$ is a real-valued function, and ∂_{x_j} stands for the partial derivative with respect to x_j. The parameters δ, χ, b, and m are real constants. Of course, the summation in the second equation does not exist if $n = 2$.

In case $n = 2$ and $\rho = 2$, system (4.3) was derived by Davey and Stewartson [14] as a model for the evolution of weakly nonlinear packets of water waves that travel predominantly in one direction, but for which the amplitude of the waves is modulated in two directions. From the physical point of view, u represents the short-wave and φ represents the long-wave amplitude.

System (4.3) was classified by Ghidaglia and Saut [17] in the two dimensional case as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic, and hyperbolic-hyperbolic according to the sign of (δ, m): $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$. For the sake of convenience, we adopt the same classification here, even for $n \geq 3$ and $\rho \neq 2$.

The IVP associated with (4.3) has gained the attention of a broad community of researchers in the past few years. As far as we know, it has started with [18]. In particular, by considering the elliptic-elliptic and hyperbolic-elliptic cases, the authors established local well-posedness in $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, and $H^2(\mathbb{R}^2)$. Global and blow-up results were also proved. Moreover, global existence for small data were established in the elliptic-hyperbolic case (see also [21], [22], [23], [24], [25], and [28]).

In the elliptic-elliptic case, existence of infinite energy solutions was addressed in [3] and [43]. By using interpolation spaces and following [6], the author in [3] proved the existence of global solutions, in weak Lebesgue spaces, for dimensions 2 and 3. As a result, the existence of self-similar solutions was also proved. In [43], the author used the ideas of [7] to prove the existence of self-similar solutions.

The elliptic-elliptic and hyperbolic-elliptic cases are easier to deal with because, at least formally, they can be reduced to a nonlinear Schrödinger equation as in (1.1). Indeed, by using the Fourier transform we see that (4.3) reduces to (1.1) with

$$Lu = \delta \partial_{x_1}^2 u + \sum_{j=2}^{n} \partial_{x_j}^2 u,$$

and E defined through its Fourier transform by

$$\widehat{E(f)}(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^{n} \xi_j^2} \widehat{f}(\xi). \tag{4.4}$$

Since

$$q(\xi) = \delta\xi_1^2 + \sum_{j=2}^{n} \xi_j^2,$$

we have $d = 2$, and the assumptions (H1) and (H2) hold. Moreover, as is well-known, by the Mihlin–Hörmander multiplier theorem, E is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. The real interpolation method (see e.g. [30, Theorem 3.1]) then gives us that E is also bounded from $L^{(p,\infty)}$ to $L^{(p,\infty)}$ for any $1 < p < \infty$. Thus, (H3) is also satisfied. Consequently, the results presented in Section 3 do apply to the DS system (4.3) in the elliptic-elliptic and hyperbolic-elliptic cases. Note that according to Remark 3.2, Theorem 3.1 only applies if $n = 2$.

4.3. **A generalized Davey–Stewartson system.** We also consider the generalized two-dimensional Davey–Stewartson system

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \partial_{x_2}^2 u = \chi|u|^\rho u + bu(\partial_{x_1}\varphi + \partial_{x_2}\phi), \\ \partial_{x_1}^2\varphi + m_2\partial_{x_2}^2\varphi + \ell\partial_{x_1x_2}^2\phi = \partial_{x_1}(|u|^\rho), \\ \lambda\partial_{x_1}^2\phi + m_1\partial_{x_2}^2\phi + \ell\partial_{x_1x_2}^2\varphi = \partial_{x_2}(|u|^\rho), \end{cases} \tag{4.5}$$

where $(x, t) \in \mathbb{R}^2 \times \mathbb{R},\ x = (x_1, x_2)$, $u = u(x, t)$ is a complex-valued function, and $\varphi = \varphi(x, t)$ and $\phi = \phi(x, t)$ are real-valued functions. The parameters δ, χ, and b are real constants, and m_1, m_2, λ, and ℓ satisfy the "fundamental" relation

$$(\lambda - 1)(m_2 - m_1) = \ell^2. \tag{4.6}$$

For $\rho = 2$, system (4.5) was derived quite recently in [2], and it is a model to describe the wave propagation in a bulk medium composed of an elastic material with couple stresses.

A similar classification as that for (4.3) according the sign of (m_1, m_2, λ) can be made (see [1]).

Under suitable conditions on the parameters, global existence in $H^1(\mathbb{R}^n)$ and existence/nonexistence of ground states, as well as their orbital stability, were studied in [1].

To write (4.5) in form (1.1), one first takes the Fourier transform in the second and third equations of (4.5) to get

$$\begin{cases} -(\xi_1^2 + m_2\xi_2^2)\widehat{\varphi}(\xi) - \ell\xi_1\xi_2\widehat{\phi}(\xi) = i\xi_1(|u|^\rho)^\wedge(\xi), \\ -(\lambda\xi_1^2 + m_1\xi_2^2)\widehat{\phi}(\xi) - \ell\xi_1\xi_2\widehat{\varphi}(\xi) = i\xi_2(|u|^\rho)^\wedge(\xi). \end{cases} \tag{4.7}$$

Equations (4.7) can be seen as an algebraic system in $\widehat{\phi}(\xi)$ and $\widehat{\varphi}(\xi)$. Thus, if

$$\begin{aligned}\Lambda(\xi_1,\xi_2) &= (\xi_1^2+m_2\xi_2^2)(\lambda\xi_1^2+m_1\xi_2^2)-\ell^2\xi_1^2\xi_2^2 \\ &= (\xi_1^2+m_1\xi_2^2)(\lambda\xi_1^2+m_2\xi_2^2)\neq 0,\end{aligned} \tag{4.8}$$

for $(\xi_1,\xi_2)\neq(0,0)$, we can solve (4.7) and obtain

$$\widehat{\varphi}(\xi)=\frac{i\xi_1}{\Lambda(\xi_1,\xi_2)}(\ell\xi_2^2-\lambda\xi_1^2-m_1\xi_2^2)(|u|^\rho)^\wedge(\xi), \tag{4.9}$$

$$\widehat{\phi}(\xi)=\frac{i\xi_2}{\Lambda(\xi_1,\xi_2)}(\ell\xi_1^2-\xi_1^2-m_2\xi_2^2)(|u|^\rho)^\wedge(\xi). \tag{4.10}$$

Note that in (4.8) we have used (4.6). Now, substituting (4.9) and (4.10) in the first equation of (4.5), we face the equation

$$i\partial_t u+\delta\partial_{x_1}^2u+\partial_{x_2}^2u=\chi|u|^\rho u+buE(|u|^\rho), \tag{4.11}$$

with E defined by

$$\widehat{E(f)}(\xi)=\frac{\lambda\xi_1^4+(1+m_1-2\ell)\xi_1^2\xi_2^2+m_2\xi_2^4}{\Lambda(\xi_1,\xi_2)}\widehat{f}(\xi)=e(\xi)\widehat{f}(\xi). \tag{4.12}$$

Remark 4.1. To obtain that $\Lambda(\xi_1,\xi_2)\neq 0$, for $(\xi_1,\xi_2)\neq(0,0)$ it suffices to assume $m_1>0$ and either $m_2>0$ and $\lambda>0$, or $m_2<0$ and $\lambda<0$.

Since $q(\xi)=\delta\xi_1^2+\xi_2^2$, we have $d=2$ again, and the assumptions (H1) and (H2) are satisfied. Moreover, since $e(\xi)$ is homogeneous of degree 0, the Calderon–Zygmund theory applies, and $E: L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)$ is bounded for any $1<p<\infty$ (see also [1, page 11543]). Once again, the real interpolation method implies that $E: L^{(p,\infty)}\to L^{(p,\infty)}$ is bounded for any $1<p<\infty$.

4.4. **The Grimshaw system.** Let us now consider the following two dimensional equation:

$$i\partial_t u+\lambda_1\partial_{x_1}^2u+\lambda_2\partial_{x_2}^2u=\chi|u|^\rho u+uS(u), \tag{4.13}$$

where $S=S_1+S_2$, with

$$\begin{gathered}\Delta S_2=\nu_0\partial_{x_2}^2(|u|^\rho),\\ S_1=\sum_{s=1}^\infty\delta_sW_s,\\ \Delta W_s-\gamma_s\partial_{x_1}^2W_s=\widetilde{\nu}_s\partial_{x_2}^2(|u|^\rho),\end{gathered}$$

where $(x,t)\in\mathbb{R}^2\times\mathbb{R}$, $x=(x_1,x_2)$, and $u=u(x,t)$ is a complex-valued function. For $\rho=2$, (4.13) was deduced by Grimshaw (see [15, page 257]), and it describes the amplitude of the vertical component of the velocity of an inviscid, incompressible, stratified fluid occupying a horizontal channel along which an internal gravity-wave packet is propagating (see also [18]). The parameters λ_1, λ_2, χ, ν_0, γ_s, δ_s, and $\widetilde{\nu}_s$ are all real constants with $\delta_s > 0$ and $\gamma_s > 0$.

By assuming that $\gamma_s < 1$, $s = 1, 2, \ldots$, we can rewrite (4.13) as (1.1), with $E = E_1 + E_2$, and

$$\widehat{E_2(f)}(\xi) = \nu_0 \frac{\xi_2^2}{\xi_1^2 + \xi_2^2} \widehat{f}(\xi),$$

$$E_1 = \sum_{s=1}^{\infty} \delta_s W_s, \qquad \widehat{W_s(f)}(\xi) = \widetilde{\nu}_s \frac{\xi_2^2}{(1-\gamma_s)\xi_1^2 + \xi_2^2} \widehat{f}(\xi).$$

Under such conditions, (4.13) has behavior similar to the Davey–Stewartson system (4.3). In particular, the results of [17] also apply here, in order to obtain finite-energy solutions.

Note we have $d = 2$ and the assumptions (H1) and (H2) are fulfilled. The Calderon–Zygmund theory can be applied in order to show that E_2 and W_s, $s = 1, 2 \ldots$, are bounded operators from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$, $1 < p < \infty$. In addition, the norm of W_s is bounded by $\widetilde{\nu}_s$. If we assume that

$$\sum_{s=1}^{\infty} \delta_s \widetilde{\nu}_s < \infty,$$

we promptly see that E_1 is also bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$, $1 < p < \infty$. The real interpolation method then gives the hypotheses (H3).

4.5. **The Shrira system.** Here we consider the three-dimensional model

$$\begin{cases} i\partial_t u + \dfrac{\omega_{kk}}{2}\partial_x^2 u + \dfrac{\omega_{\ell\ell}}{2}\partial_y^2 u + \dfrac{\omega_{nn}}{2}\partial_z^2 u + \omega_{nk}\partial_{xz}^2 u = -uQ, \\ \partial_x^2 Q + \partial_y^2 Q = \nu \partial_y^2 |u|^\rho, \end{cases} \tag{4.14}$$

where $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, $x = (x, y, z)$, and $u = u(x, t)$ is a complex-valued function. For $\rho = 2$, (4.14) was derived by Shrira (see [40, page 132]), and it models the evolution of a three-dimensional packet of weakly nonlinear internal gravity waves propagating obliquely at an arbitrary angle to the vertical. The real parameters ω_{kk}, $\omega_{\ell\ell}$, ω_{nn}, ω_{nk}, and ν can be calculated in terms of the wave vector of the main wave and the angle between the x-axis and the wave vector. Moreover, the following non-degeneracy condition is assumed:

$$\omega_{ll}(\omega_{kk}\omega_{nn} - \omega_{nk}) \neq 0, \quad \omega_{kk} \neq 0, \quad \omega_{nn} \neq 0.$$

We can rewrite (4.14) as an equation of the form (1.1) by solving the second equation. It turns out that E is defined by

$$\widehat{E(f)}(\xi) = \nu \frac{\xi_2^2}{\xi_1^2+\xi_2^2}\widehat{f}(\xi), \qquad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

Local well-posedness of the Cauchy problem associated with (4.14), in $H^1(\mathbb{R}^3)$, may be obtained in [17, Theorem 2.2].

Once again we have $d = 2$, and assumptions (H1) and (H2) hold. The Calderon–Zygmund theory can not be applied to this situation. However, by using interpolation with BMO and Hardy spaces, one can still prove that $E : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is bounded for any $1 < p < \infty$ (see [18, page 184]). The real interpolation method gives that $E : L^{(p,\infty)} \to L^{(p,\infty)}$ is bounded for any $1 < p < \infty$.

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