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Local well-posedness for the nonlocal derivative nonlinear Schrödinger equation in Besov spaces

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ABSTRACT

In this paper we study the Cauchy problem associated with the one-dimensional integro-differential nonlocal derivative nonlinear Schrödinger equation in the Besov space $B_2^{\frac{1}{2},1}(\mathbb{R})$. The local well-posedness for small initial data in $B_2^{\frac{1}{2},1}(\mathbb{R})$ is established. Our method of proof combines the contraction principle applied to the associated integral equation together with interpolations of some smoothing effects (Kato's smoothing effects, Strichartz estimate and estimates for the maximal function) for phase localized functions associated to the linear dispersive part of the equation, and a fractional vector-valued Leibniz's rule derived by Molinet and Ribaud in (2004).

1. Introduction

This paper is concerned with the initial value problem (IVP) associated with the one-dimensional integro-differential nonlocal derivative nonlinear Schrödinger equation (INL-NLS)

$$\begin{cases} \partial_t u + i\alpha\partial_x^2 u = 2\beta u P_+\partial_x(|u|^2) - i\beta k u\mathcal{L}_h(|u|^2) + i\gamma|u|^2 u, & (x,t)\in\mathbb{R}\times\mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $u = u(x,t)$ is a complex-valued function, $k \in \{0,1\}$, α, β and γ are real non-negative parameters with $\alpha \neq 0$, $\beta \neq 0$, $P_+f = \mathcal{F}^{-1}\{\chi_{[0,+\infty)}(\cdot)\widehat{f}(\cdot)\} = \frac{1}{2}(1+i\mathcal{H})f$ and $\mathcal{L}_h$ is the operator defined by

$$\mathcal{L}_h f = (\mathcal{H} - \mathcal{T}_h)\partial_x f, \tag{2}$$

with $\mathcal{H}$ to denote the Hilbert transform defined via Fourier transform by

$$(\mathcal{H}f)^\wedge(\xi) = -i\mathrm{sgn}(\xi)\widehat{f}(\xi)$$

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and $\mathcal{T}_h$ denotes the singular integral operator

$$\mathcal{T}_h f(x) = \frac{1}{2h}\text{p.v.}\int_{-\infty}^{+\infty} \coth\left(\frac{\pi(x-y)}{2h}\right) f(y)dy,$$

where p.v. is the Cauchy principal value of the integral and $0 < h \leq \infty$.

Note that $\lim_{h\to\infty} \mathcal{L}_h f = 0$ almost everywhere, that is, $\lim_{h\to\infty} \mathcal{T}_h f = \mathcal{H} f$ a.e..

This general model was proposed by D. Pelinovsky and R. Grimshaw [20–22] to study the evolution of quasi-harmonic wave packets at the interface of a two-layer system, where the upper layer is shallow and the lower one is deep if compared to length scale of quasi-harmonic wave packets. The parameter h is proportional to the depth of the fluid. If one of the fluids is infinitely deep, i.e., $h \to \infty$ ($k = 0$), we have the nonlocal nonlinear Schrödinger equation (NL-NLS)

$$\partial_t u + i\alpha\partial_x^2 u = 2\beta u P_+ \partial_x(|u|^2) + i\gamma|u|^2 u, \quad (x,t) \in \mathbb{R}\times\mathbb{R}, \tag{3}$$

while with both fluids of finite depth ($k = 1$), the equation in (1) is the intermediate nonlocal nonlinear Schrödinger one (INL-NLS)

$$\partial_t u + i\alpha\partial_x^2 u = \beta u(1 + i\mathcal{T}_h)\partial_x(|u|^2) + i\gamma|u|^2 u, \quad (x,t) \in \mathbb{R}\times\mathbb{R}. \tag{4}$$

In [20] it was showed that Eqs. (3) and (4) with $\gamma = 0$ are integrable. Their inverse scattering was constructed in [21]. Several other physical properties of these equations have been studied by Matsuno [8–14].

For u solution of (1) and

$$v = v(x,t) = \exp\left(i\frac{\beta}{2\alpha}\int_{-\infty}^{x} |u(y,t)|^2 dy\right) u(x,t), \tag{5}$$

the following quantities are conserved:

$$Q(v) = \int_{\mathbb{R}} |v|^2 dx, \tag{6}$$

$$E(v) = \int_{\mathbb{R}}\left\{\alpha|\partial_x v|^2 + \frac{\beta}{2}|v|^2\mathcal{H}\partial_x(|v|^2) - \frac{\beta k}{2}|v|^2\mathcal{L}_h(|v|^2) + \frac{\beta^2}{4\alpha}|v|^6 + \frac{\gamma}{2}|v|^4\right\}dx. \tag{7}$$

Then, for the original equation we have the following conserved quantities along the flow of (1):

$$Q(u) = \int_{\mathbb{R}} |u|^2 dx,$$

$$E(u) = \int_{\mathbb{R}}\left\{\alpha|\partial_x u|^2 - \beta|u|^2 Im(u\partial_x\bar{u}) + \frac{\beta}{2}|u|^2\mathcal{H}\partial_x(|u|^2) - \frac{\beta k}{2}|u|^2\mathcal{L}_h(|u|^2) + \frac{\beta^2}{2\alpha}|u|^6 + \frac{\gamma}{2}|u|^4\right\}dx.$$

From the mathematical viewpoint, a few works are available in the current literature for the NL-NLS and INL-NLS equations. As far as we know, the only works concerning well-posedness for (1) are due to Angulo/Moura [2], Moura [17] and Moura/Pilod [18]. In [17], R. Moura showed that the IVP associated to (1) is locally well-posed for small initial data in Sobolev spaces $H^s(\mathbb{R})$ for $s \geq 1$, and taking advantage of the quantities (6) and (7) with $\alpha = \beta = 1$, the solution extends globally in time for initial data in $H^1(\mathbb{R})$. In [2], Angulo/Moura give a rigorous proof that a Picard interaction scheme can not be applied for solving the Cauchy problem (1) with data in Sobolev spaces of negative index, study the asymptotic behavior of solution with respect to spatial variable and also establish the nonexistence of standing wave solutions. In [18], Moura/Pilod improved the local well-posedness (LWP) theory obtained in [17] without assuming any restriction on the initial data. More precisely, it was proved the local well-posedness with initial data in the Sobolev space $H^s(\mathbb{R})$ for $s > \frac{1}{2}$.

Our goal here is to prove LWP for small initial data in the inhomogeneous Besov space $B_2^{\frac{1}{2},1}(\mathbb{R})$. Throughout the paper, by well-posedness we mean existence, uniqueness, persistence property, and continuous dependence upon the initial data. Moreover, by a solution we mean a solution in the sense of the associated integral equation. Our result is the following (for the definitions of the space X_T see notation (27)).

Theorem 1.1. *There exists $\delta > 0$ such that for any $u_0 \in B_2^{\frac{1}{2},1}(\mathbb{R})$, with $\|u_0\|_{B_2^{\frac{1}{2},1}} < \delta$, there exist a positive time $T = T(\|u_0\|_{B_2^{\frac{1}{2},1}})$ with $T(\|u_0\|_{B_2^{\frac{1}{2},1}}) \to \infty$ when $\|u_0\|_{B_2^{\frac{1}{2},1}} \to 0$, a space X_T such that $X_T \hookrightarrow C([-T,T]; B_2^{\frac{1}{2},1}(\mathbb{R}))$, and a unique solution u to the Cauchy problem (1) in X_T. Furthermore, for any $T' \in (0,T)$ there exists $\epsilon > 0$ such that the flow-map data-solution is Lipschitz from $\{\widetilde{u}_0 \in B_2^{\frac{1}{2},1}(\mathbb{R});\ \|\widetilde{u}_0 - u_0\|_{B_2^{\frac{1}{2},1}} < \epsilon\}$ into $X_{T'}$.*

It is known that the main difficulty when one deals with equations containing derivatives in the nonlinear term, is to overcome the so-called loss of derivatives. For the equation in (1), the terms $2uP_+\partial_x(|u|^2)$ and $u(1+i\mathcal{T}_h)\partial_x(|u|^2)$ impose an additional obstacle, since the operator $\mathcal{T}_h$ (including the Hilbert transform $\mathcal{H}$) is skew-adjoint we are not allowed to perform a gauge transformation to remove the derivatives on the nonlinearity as occurs with the derivative nonlinear Schrödinger equation (8) with $\mu = 0$. We also remark that the approach based on Kato's smoothing effect and maximal estimate for the linear operator as that performed by Kenig, Ponce and Vega in [7] seems to provide well-posedness only for initial data $H^s(\mathbb{R})$ with $s > 1/2$. But such a result was already reached in [18]. The lack of maximal estimate in Sobolev spaces H^s with $s \le 1/2$ motivated us to choose Besov spaces rather than Sobolev spaces.

In [24], Takaoka proved that the IVP associated to the more general derivative Schrödinger equation

$$i\partial_t u + \partial_x^2 u = i\lambda |u|^2 \partial_x u + i\mu u^2 \partial_x \overline{u}, \tag{8}$$

where $u = u(x,t)$ is a complex-valued function, and λ and μ are two complex constants, is locally well-posed in $H^s(\mathbb{R})$ with $s \ge \frac{1}{2}$. He used the same gauge transformation as in [19] to cancel the term $\lambda|u|^2\partial_x u$ in the nonlinearity and the Fourier restriction norm method, developed by Bourgain and Kenig, Ponce and Vega, to handle the term $\mu u^2 \partial_x \overline{u}$.

It is worth noticing that the methods employed by Takaoka [24] do not seem to apply in the case of the IVPs associated to Eqs. (3) and (4), since those equations are gauge equivalent to

$$\partial_t v + i\alpha\partial_x^2 v = i\frac{3\beta^2}{4\alpha}|v|^4 v + i\beta v\mathcal{T}_h\partial_x(|v|^2) + i\gamma|v|^2 v, \quad 0 < h \le \infty, \tag{9}$$

via the change of variable (5) and, as observed in the introduction of [24], the Fourier restriction norm method seems inapplicable to the nonlinearity $|v|^2\partial_x v$, appearing implicitly in the nonlinear part of (9).

In order to prove our result we follow the recent approach introduced by Molinet and Ribaud for the generalized Benjamin–Ono equation [16]. Our strategy combines the contraction principle applied to the associated integral equation

$$u(t) = U(t)u_0 + 2\int_0^t U(t-s)(2\beta u P_+\partial_x(|u|^2) - i\beta k u\mathcal{L}_h(|u|^2) + i\gamma|u|^2 u)(s)ds \tag{10}$$

together with interpolations of some linear estimates (Kato's smoothing effects, Strichartz estimate and estimates for the maximal function) for phase localized functions associated to the linear dispersive part of the equation, and a fractional vector-valued Leibniz's rule derived by Molinet and Ribaud in [15].

The work is organized as follows. In Section 2 we present the notation, auxiliary lemmas and the resolution space. In Section 3 some linear estimates determined by the group $\{e^{-it\partial_x^2}\}_{t=-\infty}^{\infty}$ for phase localized functions are deduced from their corresponding non-localized version. We finish with nonlinear estimates and the proof of Theorem 1.1.

2. Notation, auxiliary lemmas and the resolution space

Given any positives constants C, D, by $C \lesssim D$ we mean that there exists a constant $c > 0$ such that $C \leq cD$; and, by $C \sim D$ we mean $C \lesssim D$ and $D \lesssim C$. Given two operators A and B, we denote by $[A,B] = AB - BA$ the commutator between A and B. By $\mathcal{F}\{u\}$ or $\widehat{u}$ we will denote the Fourier transform of u with respect to the space variable x, while $\mathcal{F}^{-1}\{u\}$ or $\check{u}$ will denote its inverse Fourier transform. L^p-norms will be written as $\|\cdot\|_{L^p_x}$ or $\|\cdot\|_{L^p}$ if no confusion is caused. For $1 \leq p, q < \infty$ and $f : \mathbb{R} \times [0,T] \to \mathbb{R}$, we define

$$\|f\|_{L^p_x L^q_T} = \|\, \|f(\cdot,\cdot)\,\|_{L^q_T}\|_{L^p_x}\,.$$

$\|f\|_{L^q_T L^p_x}$ is similarly defined, and when $p = \infty$ or $q = \infty$, $\|f\|_{L^p_x L^q_T}$ is defined in the natural form. When $p = q$, we will write $\|f\|_{L^p_x L^q_T}$ as $\|f\|_{L^p_{x,T}}$.

$\mathcal{S}(\mathbb{R})$ will represent the Schwartz space. For $s \in \mathbb{R}$ (and $f \in \mathcal{S}'$) $J^s = \mathcal{F}^{-1}\Big((1+|\cdot|^2)^{\frac{s}{2}}\widehat{f}\Big)$ will be the Bessel potential of order $-s$, $D^s_x f = \mathcal{F}^{-1}\Big(|\cdot|^s\widehat{f}\Big)$ denotes the Riesz potential of order $-s$, and $\widetilde{D}^s_x = \mathcal{H}D^s_x$. The space $H^s(\mathbb{R})$ is the usual Sobolev space with norm $\|\cdot\|_{H^s} := \|J^s \cdot\|_{L^2}$.

Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that, $0 \leq \psi \leq 1$, $\psi(\xi) = 1$ for $|\xi| \leq 1$, and $\psi(\xi) = 0$ for $|\xi| > 2$. We define

$$\varphi(\xi) = \psi(\xi) - \psi(2\xi), \qquad \varphi_j(\xi) = \varphi(2^{-j}\xi) \quad (j \in \mathbb{Z}),$$

so that

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \;\; \xi \neq 0, \quad \text{and} \quad \operatorname{supp}(\varphi_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}.$$

We note that $\psi(\xi) = 1 - \sum_{j\geq 1}\varphi_j(\xi)$.

Next, we define the Littlewood–Paley multiplier as

$$\Delta_j f = (\varphi_j \widehat{f})^\vee = \overset{\vee}{\varphi}_j * f, \quad \text{and} \quad S_j f = \sum_{k\leq j} \Delta_k f, \;\; \text{with } f \in \mathcal{S}', \quad j \in \mathbb{Z}. \tag{11}$$

Note that

$$S_0 f = (\psi \widehat{f})^\vee, \;\; \forall\, f \in \mathcal{S}'(\mathbb{R}^n), \tag{12}$$

(S_0 is the projection in low frequency)

$$\textit{supp}\, \widehat{S_j f} \subseteq \{\, \xi \,:\, |\xi| \leq 2^{j+1}\,\} \text{ and for } |\xi| \leq 2^j, \;\; \widehat{S_j f}(\xi) = \widehat{f}(\xi), \tag{13}$$

$$\textit{supp}\, \widehat{\triangle_j f} \subseteq \{\, \xi \,:\, 2^{j-1} \leq |\xi| \leq 2^{j+1}\,\}, \tag{14}$$

$$f = \sum_{j\in\mathbb{Z}} \triangle_j f = S_0 f + \sum_{j\geq 1} \Delta_j f = S_0 f + P_{hig} f, \tag{15}$$

$$\begin{aligned} fg &= S_0 f S_0 g + \sum_{r\geq 0}(S_{r+1} f S_{r+1} g - S_r f S_r g) \\ &= S_0 f S_0 g + \sum_{r\geq 0}(\Delta_{r+1} f S_r g + \Delta_{r+1} g S_{r+1} f), \end{aligned} \tag{16}$$

$$\Delta_j(S_0 f S_0 g) = 0, \;\text{for all } j \geq 3, \tag{17}$$

and,

$$\Delta_j\{\sum_{r\geq 0}(\Delta_{r+1} f S_r g + \Delta_{r+1} g S_{r+1} f)\} = \Delta_j\{\sum_{r\geq j}(\Delta_{r-2} f S_{r-3} g + \Delta_{r-2} g S_{r-2} f)\}. \tag{18}$$

If we define $\widetilde{\Delta}_j := \sum_{k=-1}^{1} \Delta_{j+k}$, then we have that

$$\widetilde{\Delta}_j \circ \Delta_j = \Delta_j. \tag{19}$$

Let C_r, $r \in \mathbb{N}$, be positive real constants. Then

$$\sum_{j\geq 1}\sum_{r\geq j} 2^j C_r \leq \sum_{r\geq 0} 2^r C_r. \tag{20}$$

In fact,

$$\begin{aligned}\sum_{j\geq 1}\sum_{r\geq j} 2^j C_r &= \sum_{j\geq 1}\sum_{r\geq 0} \chi_{\leq 0}(j-r)2^{j-r}2^r C_r = \sum_{r\geq 0} 2^r C_r \sum_{j\geq 1} \chi_{\leq 0}(j-r)2^{j-r}\\ &\leq \sum_{r\geq 0} 2^r C_r \sum_{l\geq 0} 2^{-l} = 2\sum_{r\geq 0} 2^r C_r.\end{aligned}$$

We will denote by $P_+ f = \mathcal{F}^{-1}\{\chi_{[0,+\infty)}(\cdot)\widehat{f}(\cdot)\}$ and $P_- f = \mathcal{F}^{-1}\{\chi_{(-\infty,0]}(\cdot)\widehat{f}(\cdot)\}$ the projection in positive and negative frequencies of f, respectively. We also define

$$P_{hig} f = (1-S_0)f, \quad P_{+hig} f = P_+ P_{hig} f \quad \text{and} \quad P_{-hig} f = P_- P_{hig} f.$$

It is well known that P_{hig} and $P_{\pm hig}$ are continuous operators on $L^p_x L^q_T$, for any $1 \leq p, q \leq \infty$. From the definitions of S_0 and P_{hig}, it follows that $P_{hig} \circ S_0 = 0$.

In the next lemma we state some results about Littlewood–Paley multipliers whose proof can be found in [3], Lemma 6.2.1, page 140:

Lemma 2.1. *Let $1 \leq p \leq \infty$ and $f \in \mathcal{S}'$ be such that $\triangle_j f \in L^p$. Then for all $s \in \mathbb{R}$,*
(i) $\|J^s \triangle_j f\|_p \leq c2^{sj}\| \triangle_j f\|_p, \ \ \forall\, j \geq 1$,
(ii) $\|D^s \triangle_j f\|_p \leq c2^{sj}\| \triangle_j f\|_p, \ \ \forall\, j \in \mathbb{Z}$,
(iii) $\|J^s S_0 f\|_p \leq c\|S_0 f\|_p$,
with c independent of p and j.

We also have the following version of estimate (ii) in Lemma 2.1 for mixed spaces.

Lemma 2.2. *Let $f : \mathbb{R}\times[0,T] \longrightarrow \mathbb{C}$ $(0 < T < \infty)$ be a smooth function and $p, q \in [1,\infty]$. Then for any $j \in \mathbb{N}$,*

$$\|D^s_x \Delta_j f\|_{L^p_x L^q_T} \lesssim 2^{js}\|\Delta_j f\|_{L^p_x L^q_T} \ \ (s \in \mathbb{R}). \tag{21}$$

The estimate (21) also holds with ∂^k_x $(k \in \mathbb{N})$ in place of D^s_x.

Proof. We follow the proof of Lemma 2.2 in [23], in which the estimate (21) is proven for ∂^k_x, $k \in \mathbb{N}$. From (19), Minkowski and Young inequalities we have

$$\begin{aligned}\| \triangle_j D^s_x f\|_{L^p_x L^q_T} &= \|\widetilde{\triangle}_j \triangle_j D^s_x f\|_{L^p_x L^q_T} = \|D^s_x(\widetilde{\varphi}_j)^{\vee} * \triangle_j f\|_{L^p_x L^q_T}\\ &\lesssim \|D^s_x \check{\widetilde{\varphi}}_j\|_{L^1_x} \| \triangle_j f\|_{L^p_x L^q_T}.\end{aligned}$$

Since $\widetilde{\varphi} \in \mathcal{S}$, we get

$$\begin{aligned}\|D^s_x \check{\widetilde{\varphi}}_j\|_{L^1_x} &= 2^{js}\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i2^j x\cdot\eta} \mid \eta \mid^s \tilde{\varphi}(\eta)d\eta \right| 2^j dx = 2^{js}\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{iy\cdot\eta} \mid \eta \mid^s \tilde{\varphi}(\eta)d\eta \right| dy\\ &= 2^{js}\|D^s \check{\tilde{\varphi}}\|_{L^1} \leq C2^{js},\end{aligned}$$

and thus, the result is proven. □

The following commutator estimate will be important to reach our results.

Lemma 2.3. *Let $\alpha \in (0,1)$ and $0 \leq \beta < 1-\alpha$. If $p,\ p_1,\ p_2,\ q,\ q_1,\ q_2 \in (1,\infty)$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*

$$\|D_x^\beta([D_x^\alpha, f]g)\|_{L_x^p L_T^q} \lesssim \|g\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha+\beta} f\|_{L_x^{p_2} L_T^{q_2}}.$$

Moreover, the value $q_1 = \infty$ is guaranteed when $\beta > 0$. The lemma is still valid with $\widetilde{D}_x^\alpha$, $P_+ D_x^\alpha$ or $P_+ \widetilde{D}_x^\alpha$ in place of D_x^α.

Proof. See Lema 3.7 in Molinet and Ribaud [15]. □

Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The Besov spaces $B_p^{s,q}(\mathbb{R})$ are defined by

$$B_p^{s,q} = \{f \in \mathcal{S}'(\mathbb{R});\ \|f\|_{B_p^{s,q}} = \|S_0 f\|_{L^p} + \big(\sum_{j\geq 1} 2^{qjs} \|\Delta_j f\|_{L^p}^q\big)^{\frac{1}{q}} < \infty\}.$$

The homogeneous Besov spaces are defined as

$$\dot{B}_p^{s,q} = \{f \in \mathcal{S}'(\mathbb{R});\ \|f\|_{\dot{B}_p^{s,q}} = \big(\sum_{j\in\mathbb{Z}} 2^{qjs} \|\Delta_j f\|_{L^p}^q\big)^{\frac{1}{q}} < \infty\}.$$

In the next lemma, we list some facts about Besov spaces and homogeneous Besov spaces. For the proofs we refer the reader to [3]:

Lemma 2.4. *Let s, s_1, s_2 be real numbers. Then the following properties hold:*
(i) If $s_1 < s_2$ then $\|f\|_{B_p^{s_1,q}} \leq \|f\|_{B_p^{s_2,q}}$;
(ii) If $0 \notin supp\,\widehat{f}$ then $\|f\|_{\dot{B}_p^{s,q}} \simeq \|f\|_{B_p^{s,q}}$,
(iii) If $s = \frac{1}{p} - \frac{1}{r}$ then $\|f\|_{L^r} \leq \|f\|_{B_p^{s,1}}$.

Proof. For the proof of (i) see Theorem 6.2.4 in [3]. The proof of (ii) can be found in [3], Theorem 6.3.2. For item (iii) see the proof of Theorem 6.5.1 in [3]. □

We also need the following estimate for the operator $\mathcal{L}_h$ defined in (2).

Lemma 2.5. *There exists $C = C(h) > 0$ such that*

$$\|\mathcal{L}_h f\|_{B_2^{\frac{1}{2},1}} \leq C \|f\|_{B_2^{\frac{1}{2},1}}, \tag{22}$$

for any $f \in B_2^{\frac{1}{2},1}(\mathbb{R})$.

Proof. Expanding in series the term $\coth(h\xi)$ of $\widehat{\mathcal{L}_h f}(\xi) = (|\xi| - \xi\coth(h\xi) + \frac{1}{h})\widehat{f}(\xi)$, we get

$$0 \leq |\xi| - \xi\coth(h\xi) + \frac{1}{h} \leq \frac{2}{h},$$

which implies (22). For more details, we address the reader to reference [1], Lemma 4.1. □

We finish this section defining our space of resolution. Given $T > 0$, we define the following semi norms:

$$N_1(u) = \|S_0 u\|_{L_T^\infty L_x^2} + \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j u\|_{L_T^\infty L_x^2}, \tag{23}$$

$$N_2(u) = \|S_0 u\|_{L^\infty_x L^2_T} + \sum_{j\geq 1} 2^j \|\Delta_j u\|_{L^\infty_x L^2_T}, \tag{24}$$

$$N_3(u) = \|S_0 u\|_{L^2_x L^\infty_T} + \sum_{j\geq 1} \|\Delta_j u\|_{L^2_x L^\infty_T}, \tag{25}$$

$$N_4(u) = \|S_0 u\|_{L^4_{x,T}} + \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j u\|_{L^4_{x,T}}, \tag{26}$$

Then let X_T be the Banach space

$$X_T = \{\, u \in C([-T,T], B_2^{\frac{1}{2},1}(\mathbb{R})); \|u\|_{X_T} < \infty\}, \quad \text{where } \|u\|_{X_T} = \sum_{m=1}^{4} N_m(u). \tag{27}$$

3. Linear estimates

In this section we present some estimates associated to the solution of the linear IVP

$$\begin{cases} \partial_t u + i\partial_x^2 u = f(x,t), \\ u(x,0) = \phi(x) \end{cases} \quad (x,t) \in \mathbb{R}\times\mathbb{R}, \tag{28}$$

Let

$$U(t)\phi(x) = \int_{\mathbb{R}} e^{i(x\xi + t\xi^2)} \widehat{\phi}(\xi) d\xi$$

be the solution to the linear homogeneous problem associated to (28) (with $f = 0$). We will also make frequent use of estimates for the retarded operator I defined by

$$I(f)(t) = \int_0^t U(t-t')f(t')dt'. \tag{29}$$

Thus the solution of (28) can be written as

$$u(t) = U(t)\phi + I(f)(t).$$

In order to simplify the notation, we employ the following definition as in Molinet and Ribaud [15].

Definition 3.1. We say that a triplet $(\alpha, p, q) \in \mathbb{R}\times[2,\infty]^2$ is
(i) 1-admissible, if and only if,

$$(\alpha,p,q) = (\frac{1}{2}, \infty, 2) \;\; or \;\; p\in[4,\infty),\ q\in[2,\infty],\ \frac{2}{p} + \frac{1}{q} \leq \frac{1}{2},\ \alpha = \frac{1}{p} + \frac{2}{q} - \frac{1}{2};$$

(ii) 2-admissible, if and only if,

$$2 < p, q < \infty, \;\; \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \;\; \alpha = \frac{1}{p} + \frac{3}{q} - 1.$$

(iii) 2^*-admissible if, and only if, it is 2-admissible and $4 \leq p < \infty$.

3.1. Linear estimates for the free and the inhomogeneous evolutions.

We first list the smoothing effects and Strichartz estimates obtained by Molinet and Ribaud in [15] interpolating previous results of Kenig, Ponce and Vega in [5,6].

Lemma 3.2. *Let be $(\alpha,\, p,\, q) \in \mathbb{R}\times[2,\infty]^2$ and $0 < T < 1$.*

(i) If $(\alpha,\, p,\, q)$ is 1-admissible, then

$$\|D_x^\alpha U(t)\phi\|_{L^p_x L^q_T} \lesssim \|\phi\|_{L^2}, \quad \forall\, \phi \in \mathcal{S}(\mathbb{R}). \tag{30}$$

(ii) If (α, p, q) is 2-admissible, then

$$\|J_x^{\alpha} U(t)\phi\|_{L_x^p L_T^q} \lesssim \|\phi\|_{L^2}, \quad \forall\, \phi \in \mathcal{S}(\mathbb{R}). \tag{31}$$

(iii) If (α, p, q) is 2^*-admissible, then

$$\|D_x^{\alpha} U(t)\phi\|_{L_x^p L_T^q} \lesssim T^{\frac{1}{4}-\frac{1}{2q}} \|\phi\|_{L^2}, \quad \forall\, \phi \in \mathcal{S}(\mathbb{R}). \tag{32}$$

Proof. See [15], Proposition 2.3. □

Note that the estimate (i) in Lemma 3.2 generalizes the classical Kato's estimate

$$\|D_x^{\frac{1}{2}} U(t)\phi\|_{L_x^\infty L_T^2} \lesssim \|\phi\|_{L^2}.$$

We also observe that Lemma 3.2 does not cover the triplets $(-s, 2, \infty)$, $s > 1/2$, since they are not admissible. These triplets were covered by Kenig, Ponce and Vega who proved the following L^2 maximal function estimate:

Lemma 3.3. *For any $s > 1/2$ and $0 < T \le 1$, it has*

$$\|U(t)\phi\|_{L_x^2 L_T^\infty} \lesssim \|\phi\|_{H^s}. \tag{33}$$

Proof. See [7], Theorem 3.1. □

Making use of the so-called Christ/Kiselev Lemma [4], Molinet and Ribaud deduced the following retarded estimates from the above nonretarded ones.

Lemma 3.4. *Let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $(r_1, r_2) \in \mathbb{R}_+^2$ and $1 \le p_1, q_1, p_2, q_2 \le \infty$ be such that, given $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\|D_x^{\alpha_1} U(t)\phi\|_{L_x^{p_1} L_T^{q_1}} \lesssim T^{r_1} \|\phi\|_{L^2}, \tag{34}$$

$$\|D_x^{\alpha_2} U(t)\phi\|_{L_x^{p_2} L_T^{q_2}} \lesssim T^{r_2} \|\phi\|_{L^2}. \tag{35}$$

Then for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\|D_x^{\alpha_2} I(f)\|_{L_T^\infty L_x^2} \lesssim T^{r_2} \|f\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}, \tag{36}$$

$$\left\|D_x^{\alpha_1+\alpha_2} I(f)\right\|_{L_x^{p_1} L_T^{q_1}} \lesssim T^{r_1+r_2} \|f\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}, \tag{37}$$

provided

$$\min(p_1, q_1) > \max(\tilde{p}_2, \tilde{q}_2) \quad \text{or} \quad (q_1 = \infty \ \text{ and } \ \tilde{p}_2, \tilde{q}_2 < \infty), \tag{38}$$

where $\tilde{p}_2$, $\tilde{q}_2$ are defined by $\dfrac{1}{\tilde{p}_2} = 1 - \dfrac{1}{p_2}$ and $\dfrac{1}{\tilde{q}_2} = 1 - \dfrac{1}{q_2}$.

Proof. See [15], Proposition 2.7. □

Corollary 3.5. *For any $0 < T \le 1$,*

$$\|D_x^{\frac{1}{2}} I(f)\|_{L_T^\infty L_x^2} \lesssim \|f\|_{L_x^1 L_T^2}.$$

Proof. It is enough to note that the triplet $(0, 4, 4)$ is 2-admissible and the triplet $(\frac{1}{2}, \infty, 2)$ is 1-admissible, and apply Lemma 3.4 with $\alpha_1 = 0$, $p_1 = q_1 = 4$, $\alpha_2 = 1/2$, $p_2 = \infty$, $q_2 = 2$. □

We close this subsection with the following estimate not covered in Lemma 3.4:

Lemma 3.6. *For any* $0 < T \leq 1$,

$$\|\partial_x I(f)\|_{L_x^\infty L_T^2} \leq \|f\|_{L_x^1 L_T^2}.$$

Proof. We refer to [7], Theorem 2.3. □

3.2. Linear estimates for phase localized functions.

Here we are going to follow the idea from Molinet and Ribaud in [16] to obtain linear estimates for phase localized functions. We begin with the version of Lemma 3.2 for phase localized functions.

Lemma 3.7. *Let* $\phi \in \mathcal{S}(\mathbb{R})$, $(\alpha, p, q) \in \mathbb{R} \times [2, \infty]^2$, $j \in \mathbb{Z}$ *and* $0 < T < 1$.

(i) if (α, p, q) *is 1-admissible or 2-admissible, then*

$$\|U(t) \triangle_j \phi\|_{L_x^p L_T^q} \lesssim 2^{-j\alpha} \| \triangle_j \phi\|_{L^2}.$$

(ii) If (α, p, q) *is 2*-admissible, then*

$$\|U(t) \triangle_j \phi\|_{L_x^p L_T^q} \lesssim T^{\frac{1}{4}-\frac{1}{2q}} 2^{-j\alpha} \| \triangle_j \phi\|_{L^2}.$$

Proof. Let $\phi \in \mathcal{S}$.

(i) Using Lemma 2.2 and the fact that *supp* $\widehat{\Delta}_j \subseteq \{ \xi : 2^{j-1} \leq |\xi| \leq 2^{j+1} \}$ we have,

$$\|U(t)\Delta_j\phi\|_{L_x^p L_T^q} = \|U(t)D_x^{-\alpha} D_x^\alpha \Delta_j \phi\|_{L_x^p L_T^q} \lesssim 2^{-j\alpha} \|D_x^\alpha U(t)\Delta_j\phi\|_{L_x^p L_T^q}.$$

Since

$$\|D_x^\alpha U(t)\Delta_j\phi\|_{L_x^p L_T^q} \sim \|J^\alpha U(t)\Delta_j\phi\|_{L_x^p L_T^q}$$

the required estimate follows from Lemma 3.2.

(ii) It follows from the same ideas of item (i). □

Now we give an estimate involving the operator S_0.

Lemma 3.8. *If* $s \geq 0$, $p \geq 2$ *and* $1 \leq q \leq p \leq \infty$, *then*

$$\|S_0 U(t) D_x^s \phi\|_{L_x^p L_T^q} \lesssim T^{1/q} \|S_0\phi\|_{L^2}.$$

Proof. From Hölder's inequality we deduce that
$\|S_0 D_x^s U(t)\phi\|_{L_x^p L_T^q} \leq T^{\frac{p-q}{p}} \|S_0 D_x^s U(t)\phi\|_{L_{x,T}^p}$.
Using now Sobolev's embedding and properties of the operators S_0 and $U(t)$ we see that,

$$\|S_0 U(t) D_x^s \phi\|_{L_{x,T}^p} \leq \|D_x^{r+s} S_0 U(t)\phi\|_{L_T^p L_x^2} \lesssim \|S_0\phi\|_{L_T^p L_x^2} \leq T^{\frac{1}{p}} \|S_0\phi\|_{L_x^2},$$

where $r > \frac{1}{2} - \frac{1}{p}$. Thus the lemma is proved. □

We now present the phase localized estimate for the maximal function in L^2.

Lemma 3.9. *Given $\phi \in \mathcal{S}(\mathbb{R})$ and $0 < T \leq 1$, it holds*

$$\|U(t) \triangle_j \phi\|_{L^2_x L^\infty_T} \lesssim 2^{\frac{j}{2}} \| \triangle_j \phi\|_{L^2_x}. \tag{39}$$

Proof. See Molinet and Ribaud [16], estimate (28) in Proposition 4. □

The phase localized version of Lemma 3.4 reads as follows.

Lemma 3.10. *Let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $(r_1, r_2) \in \mathbb{R}^2_+$ and $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ be such that, for each $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\|U(t)\Delta_j\phi\|_{L^{p_1}_x L^{q_1}_T} \lesssim T^{r_1} 2^{-j\alpha_1} \|\Delta_j\phi\|_{L^2}, \tag{40}$$

$$\|U(t)\Delta_j\phi\|_{L^{p_2}_x L^{q_2}_T} \lesssim T^{r_2} 2^{-j\alpha_2} \|\Delta_j\phi\|_{L^2}. \tag{41}$$

Then for any $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\|I(\Delta_j f)\|_{L^\infty_T L^2_x} \lesssim T^{r_2} 2^{-j\alpha_2} \|\Delta_j f\|_{L^{\tilde{p}_2}_x L^{\tilde{q}_2}_T},$$

$$\|I(\Delta_j f)\|_{L^{p_1}_x L^{q_1}_T} \lesssim T^{r_1+r_2} 2^{-j(\alpha_1+\alpha_2)} \|\Delta_j f\|_{L^{\tilde{p}_2}_x L^{\tilde{q}_2}_T},$$

provided

$$\min(p_1, q_1) > \max(\tilde{p}_2, \tilde{q}_2) \quad \text{or} \quad (q_1 = \infty \ \text{ and } \ \tilde{p}_2, \tilde{q}_2 < \infty),$$

where $\tilde{p}_2$, $\tilde{q}_2$ are defined by $\dfrac{1}{\tilde{p}_2} = 1 - \dfrac{1}{p_2}$ and $\dfrac{1}{\tilde{q}_2} = 1 - \dfrac{1}{q_2}$.

Proof. From Lemma 2.2 we have

$$\|D^{\alpha_k}_x U(t)\Delta_j\phi\|_{L^{p_k}_x L^{q_k}_T} \lesssim 2^{j\alpha_k} \|U(t)\Delta_j\phi\|_{L^{p_k}_x L^{q_k}_T}, \ k = 1, 2. \tag{42}$$

Now we use the hypothesis (40)–(41) in (42) to infer that

$$\|D^{\alpha_1}_x U(t)\Delta_j\phi\|_{L^{p_1}_x L^{q_1}_T} \lesssim T^{r_1} \|\Delta_j\phi\|_{L^2},$$

$$\|D^{\alpha_2}_x U(t)\Delta_j\phi\|_{L^{p_2}_x L^{q_2}_T} \lesssim T^{r_2} \|\Delta_j\phi\|_{L^2}.$$

Now it is enough to apply Lemma 3.4. □

Corollary 3.11. *Let $0 < T \leq 1$. Then*

$$\|I(\Delta_j f)\|_{L^\infty_T L^2_x} \lesssim 2^{-\frac{j}{2}} \|\Delta_j f\|_{L^1_x L^2_T}.$$

Proof. First we note that the triplet $(0, 4, 4)$ is 2-admissible and the triplet $(\frac{1}{2}, \infty, 2)$ is 1-admissible. Then we apply Lemma 3.7 to infer that

$$\|U(t)\Delta_j\phi\|_{L^4_{x,T}} \lesssim \|\Delta_j\phi\|_{L^2} \quad \text{and} \quad \|U(t)\Delta_j\phi\|_{L^\infty_x L^2_T} \lesssim 2^{-\frac{j}{2}} \|\Delta_j\phi\|_{L^2}.$$

Thus, the result follows from Lemma 3.10. □

To finish, we establish the phase localized version of Lemma 3.6, which is not covered by Lemma 3.10.

Lemma 3.12. *Let* $0 < T \leq 1$. *Then*

$$\|\partial_x I(\Delta_j f)\|_{L_x^\infty L_T^2} \lesssim \|\Delta_j f\|_{L_x^1 L_T^2}, \tag{43}$$

$$\|I(\Delta_j f)\|_{L_x^\infty L_T^2} \lesssim 2^{-j}\|\Delta_j f\|_{L_x^1 L_T^2}. \tag{44}$$

Proof. The estimate (43) follows from Lemma 3.6 with $\Delta_j f$ in place of f. The estimate (44) follows from the fact that $0 \notin supp\,\widehat{\Delta}_j$ joint with (43) and Lemma 2.2. □

Combining the previous estimates we conclude the following result.

Proposition 3.13. *Let* $0 < T \leq 1$. *Then*

$$\|U(t)u_0\|_{X_T} \lesssim \|u_0\|_{B_2^{\frac{1}{2},1}}. \tag{45}$$

Proof. We note that $(\frac{1}{2}, \infty, 2)$ is 1-admissible and $(0, 4, 4)$ is 2-admissible. Then the inequality (45) follows from the fact that $\{U(t)\}$ is a L^2-unitary group, combined with Lemma 3.7 for $(\alpha, p, q) = (\frac{1}{2}, \infty, 2)$ and $(\alpha, p, q) = (0, 4, 4)$, Lemmas 3.8 and 3.9. □

4. Nonlinear estimates and proof of Theorem 1.1

To study the IVP (1) we use its integral equivalent formulation (10). Without loss of generality, we are going to consider $\alpha = \beta = \gamma = 1$ and then write

$$u(t) = U(t)u_0 + I(2uP_+\partial_x(|u|^2) - iku\mathcal{L}_h(|u|^2) + i|u|^2u), \tag{46}$$

where the operator I was defined in (29). Furthermore, we shall split $I = I(2uP_+\partial_x(|u|^2) - iku\mathcal{L}_h(|u|^2) + i|u|^2u)$ as

$$I = I_1 + I_2, \quad \text{where } I_1 = I(2uP_+\partial_x(|u|^2) - iku\mathcal{L}_h(|u|^2)) \text{ and } I_2 = I(i|u|^2u).$$

4.1. Nonlinear estimates

We consider estimates for the integral equation (46) in the functional space defined in (27).

Let P denote one of the following operators: P_+ or $\mathcal{H} - \mathcal{T}_h$. Taking $f = u$ and $g = P\partial_x(|u|^2)$ in (16) and in (18), we can write

$$uP\partial_x(|u|^2) = S_0uS_0P\partial_x(|u|^2) + \sum_{r\geq 0}\left(\Delta_{r+1}uS_rP\partial_x(|u|^2) + \Delta_{r+1}P\partial_x(|u|^2)S_{r+1}u\right), \tag{47}$$

$$\begin{aligned}\Delta_j(uP\partial_x(|u|^2)) =&\Delta_j(S_0uS_0P\partial_x(|u|^2))\\ &+ \Delta_j\sum_{r\geq j}\left(\Delta_{r-2}uS_{r-3}P\partial_x(|u|^2) + \Delta_{r-2}P\partial_x(|u|^2)S_{r-2}u\right),\end{aligned} \tag{48}$$

respectively. Using again (16) and (18) with $f = u$ and $g = \bar{u}$, we can write

$$|u|^2 = |S_0u|^2 + \sum_{r\geq 0}\left(\Delta_{r+1}uS_r\bar{u} + \Delta_{r+1}\bar{u}S_{r+1}u\right), \tag{49}$$

$$\Delta_j(|u|^2) = \Delta_j(|S_0u|^2) + \Delta_j\sum_{r\geq j}\left(\Delta_{r-2}uS_{r-3}\bar{u} + \Delta_{r-2}\bar{u}S_{r-2}u\right). \tag{50}$$

Now we are ready to estimate $\|I\|_{X_T}$.

Lemma 4.1. *Let* $0 < T \leq 1$. *Then*

$$N_1(I) \lesssim T N_1(u)^3 + T^{\frac{3}{4}} N_1(u)^2 N_4(u) + T^{\frac{1}{2}} N_1(u) N_2(u) N_3(u) + T^{\frac{13}{40}} N_1(u) N_3(u) N_4(u) + T^{\frac{1}{40}} N_2(u)^{\frac{1}{5}} N_3(u) N_4(u)^{\frac{9}{5}} + N_3(u)^2 N_2(u).$$

Proof. From Lemma 3.8 we have

$$\|U(t)S_0u_0\|_{L^4_{x,T}} \lesssim T^{\frac{1}{4}} \|S_0u_0\|_{L^2}, \tag{51}$$

and then by Lemma 3.4 with $\phi = S_0u_0$, $f = S_0(uP\partial_x(|u|^2))$, $\alpha_2 = 0$, $p_2 = q_2 = 4$ and $r_2 = \frac{1}{4}$, we get

$$\|S_0I_1\|_{L^\infty_T L^2_x} \lesssim T^{\frac{1}{4}} \|S_0(uP\partial_x(|u|^2))\|_{L^{\frac{4}{3}}_{x,T}}. \tag{52}$$

Using that S_0 and P are bounded from $L^{\frac{4}{3}}_{x,T}$ to itself (see Lemma 3.1 in [17]), and then Hölder inequality, Sobolev embedding and Littlewood-Paley decomposition, we infer that

$$T^{\frac{1}{4}} \|S_0(uP\partial_x(|u|^2))\|_{L^{\frac{4}{3}}_{x,T}} \lesssim T^{\frac{1}{4}} \|u\|_{L^4_{x,T}} \|\bar{u}\partial_x u\|_{L^2_{x,T}} \lesssim T^{\frac{1}{2}} N_1(u)N_2(u)N_3(u). \tag{53}$$

From Minkowski inequalities and the boundedness of S_0 in L^2_x, we have

$$\|S_0I_2\|_{L^\infty_T L^2_x} \lesssim T\|S_0(|u|^2u)\|_{L^\infty_T L^2_x} \lesssim T\||u|^2u\|_{L^\infty_T L^2_x}.$$

Now from Lemma 2.4, we get

$$T\||u|^2u\|_{L^\infty_T L^2_x} \lesssim T\|u\|^3_{L^\infty_T L^6_x} \lesssim T\|u\|^3_{L^\infty_T B^{\frac{1}{2},1}_2} \lesssim TN_1(u)^3,$$

and then we have

$$\|S_0I_2\|_{L^\infty_T L^2_x} \lesssim TN_1(u)^3. \tag{54}$$

Therefore, by (52), (53) and (54) we get the estimate

$$\|S_0I\|_{L^\infty_T L^2_x} \lesssim TN_1(u)^3 + T^{\frac{1}{2}} N_1(u)N_2(u)N_3(u). \tag{55}$$

We are now going to estimate $\sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j I_1\|_{L^\infty_T L^2_x}$. Using identities (47) and (48), we deduce that

$$\begin{aligned}
\sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j I_1\|_{L^\infty_T L^2_x} &\lesssim \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j \int_0^t U(t-s) S_0 u S_0 P\partial_x(|u|^2) ds\|_{L^\infty_T L^2_x} \\
&\quad + \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j \int_0^t U(t-s) \sum_{r\geq j} \Delta_{r-2} u S_{r-3} P\partial_x(|u|^2) ds\|_{L^\infty_T L^2_x} \\
&\quad + \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j \int_0^t U(t-s) \sum_{r\geq j} S_{r-2} u \Delta_{r-2} P\partial_x(|u|^2) ds\|_{L^\infty_T L^2_x} \\
&:= \sum_{m=1}^{3} \sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j I_{1,m}\|_{L^\infty_T L^2_x}.
\end{aligned} \tag{56}$$

Since $S_0 P\partial_x$ is bounded in L^2, we get from (17) and Sobolev embedding the estimate

$$\sum_{j\geq 1} 2^{\frac{j}{2}} \|\Delta_j I_{1,1}\|_{L^\infty_T L^2_x} \lesssim T\|S_0 u S_0 P\partial_x(|u|^2)\|_{L^\infty_T L^2_x} \lesssim TN_1(u)^3. \tag{57}$$

In order to treat $\sum_{j\geq 1} 2^{\frac{j}{2}} \|\varDelta_j I_{1,2}\|_{L^\infty_T L^2_x}$, we first use commutators to write

$$\begin{aligned}\varDelta_{r-2} u S_{r-3} P\partial_x(|u|^2) &= -\varDelta_{r-2} u D_x^{\frac{1}{2}} S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2) \\ &= [D_x^{\frac{1}{2}}, \varDelta_{r-2}u] S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2) - D_x^{\frac{1}{2}}(\varDelta_{r-2} u S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2)).\end{aligned} \tag{58}$$

Since $(\frac{2}{5}, 20, \frac{20}{9})$ is 2^*-admissible, we have from Lemma 3.7 that

$$\|U(t)\varDelta_j u_0\|_{L^{20}_x L^{\frac{20}{9}}_T} \lesssim T^{\frac{1}{40}} 2^{-\frac{2}{5}j} \|\varDelta_j u_0\|_{L^2}. \tag{59}$$

Then using Lemma 3.10 with $(\alpha_2, p_2, q_2) = (\frac{2}{5}, 20, \frac{20}{9})$, Lemma 2.3 (with $\beta = 0$ and $\alpha = \frac{1}{2}$), identity (58) and the Hölder inequality, we deduce that

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{j}{2}} \|\varDelta_j I_{1,2}\|_{L^\infty_T L^2_x} &\lesssim T^{\frac{1}{40}} \sum_{j\geq 1} 2^{\frac{j}{10}} \sum_{r\geq j} \|\varDelta_{r-2} u S_{r-3} P\partial_x(|u|^2)\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}} \\ &\lesssim T^{\frac{1}{40}} \sum_{j\geq 1} 2^{\frac{j}{10}} \sum_{r\geq j} \|[D_x^{1/2}, \varDelta_{r-2}u] S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2)\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}} \\ &\quad + T^{\frac{1}{40}} \sum_{j\geq 1} 2^{\frac{j}{2}} \cdot 2^{\frac{j}{10}} \sum_{r\geq j} \|\varDelta_{r-2} u S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2)\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}} \\ &\lesssim T^{\frac{1}{40}} \sum_{r\geq 1} 2^{\frac{r}{10}} \cdot 2^{\frac{r}{2}} \|S_{r-3} P\widetilde{D}_x^{\frac{1}{2}}(|u|^2)\|_{L_x^{\frac{4}{3}} L_T^4} \|\varDelta_{r-2} u\|_{L^5_x L_T^{\frac{10}{3}}} \\ &\lesssim T^{\frac{1}{40}} \|D_x^{1/2} u\|_{L^4_{x,T}} \|u\|_{L^2_x L^\infty_T} \sum_{r\geq 1} 2^{\frac{3r}{5}} \|\varDelta_r u\|_{L^5_x L_T^{\frac{10}{3}}}.\end{aligned} \tag{60}$$

It remains to estimate $\sum_{r\geq 1} 2^{\frac{3r}{5}} \|\varDelta_r u\|_{L^5_x L_T^{\frac{10}{3}}}$ in (60). Given $0 \leq \theta \leq 1$, we have by interpolation that

$$\begin{aligned}\sum_{r\geq 1} 2^{\frac{1+\theta}{2} r} \|\varDelta_r u\|_{L_x^{\frac{4}{1-\theta}} L_T^{\frac{4}{1+\theta}}} &\lesssim \sum_{r\geq 1} \left(2^{\frac{r}{2}} \|\varDelta_r u\|_{L^4_{x,T}}\right)^{1-\theta} \left(2^r \|\varDelta_r u\|_{L^\infty_x L^2_T}\right)^{\theta} \\ &\lesssim \left(\sum_{r\geq 1} 2^{\frac{r}{2}} \|\varDelta_r u\|_{L^4_{x,T}}\right)^{1-\theta} \left(\sum_{r\geq 1} 2^r \|\varDelta_r u\|_{L^\infty_x L^2_T}\right)^{\theta}.\end{aligned} \tag{61}$$

Then choosing $\theta = \frac{1}{5}$ in (61) we obtain

$$\sum_{r\geq 1} 2^{\frac{3r}{5}} \|\varDelta_r u\|_{L^5_x L_T^{\frac{10}{3}}} \lesssim \left(\sum_{r\geq 1} 2^{\frac{r}{2}} \|\varDelta_r u\|_{L^4_{x,T}}\right)^{\frac{4}{5}} \left(\sum_{r\geq 1} 2^r \|\varDelta_r u\|_{L^\infty_x L^2_T}\right)^{\frac{1}{5}}. \tag{62}$$

Therefore, from (60) and (62) we get

$$\sum_{j\geq 1} 2^{\frac{j}{2}} \|\varDelta_j I_{1,2}\|_{L^\infty_T L^2_x} \lesssim T^{\frac{1}{40}} N_2(u)^{\frac{1}{5}} N_4(u)^{\frac{9}{5}} N_3(u). \tag{63}$$

Finally, we are going to estimate $\sum_{j\geq 1} 2^{\frac{j}{2}} \|\varDelta_j I_{1,3}\|_{L^\infty_T L^2_x}$. To do that we use again the commutator estimate in Lemma 2.3. First we write

$$\begin{aligned}S_{r-2} u \varDelta_{r-2} P\partial_x(|u|^2) &= -S_{r-2} u D_x^{\frac{1}{2}} P\widetilde{D}_x^{\frac{1}{2}} \varDelta_{r-2}(|u|^2) \\ &= [D_x^{\frac{1}{2}}, S_{r-2}u] P\widetilde{D}_x^{\frac{1}{2}} \varDelta_{r-2}(|u|^2) - D_x^{\frac{1}{2}}(S_{r-2} u P\widetilde{D}_x^{\frac{1}{2}} \varDelta_{r-2}(|u|^2)),\end{aligned} \tag{64}$$

and then,

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3}\|_{L^\infty_T L^2_x} &\lesssim \sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j \int_0^t U(t-s)\sum_{r\geq j}[D_x^{\frac12}, S_{r-2}u]P\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2)ds\|_{L^\infty_T L^2_x}\\ &+ \sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j \int_0^t U(t-s)\sum_{r\geq j} D_x^{\frac12}(S_{r-2}uP\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2))ds\|_{L^\infty_T L^2_x}\\ &:= \sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3,1}\|_{L^\infty_T L^2_x} + \sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3,2}\|_{L^\infty_T L^2_x}. \end{aligned} \tag{65}$$

To estimate the first term on the RHS of (65), we use Lemma 3.10 with $(\alpha_2, p_2, q_2) = (\frac{2}{5}, 20, \frac{20}{9})$, Lemma 2.3 (with $\beta = 0$ and $\alpha = \frac{1}{2}$) and Hölder inequality to deduce that

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3,1}\|_{L^\infty_T L^2_x} &\lesssim T^{\frac{1}{40}} \sum_{j\geq 1} 2^{\frac{j}{10}} \sum_{r\geq j} \|[D_x^{\frac12}, S_{r-2}u]P\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2)\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}}\\ &\lesssim T^{\frac{1}{40}} \sum_{r\geq 0} 2^{\frac{r}{10}} \|D_x^{1/2} S_{r-2}u\|_{L^4_{x,T}} \|P\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}}\\ &\lesssim T^{\frac{1}{40}} N_4(u) \sum_{r\geq 1} 2^{\frac{r}{10}} \|D_x^{\frac12}\Delta_r(|u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}}. \end{aligned} \tag{66}$$

Using decompositions (49) and (50), properties of S_r and estimate (62), we find

$$\begin{aligned}\sum_{r\geq 1} 2^{\frac{r}{10}} \|D_x^{\frac12}\Delta_r(|u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}} &\lesssim \sum_{r\geq 0} 2^{\frac{r}{10}}\cdot 2^{\frac{r}{2}}\left\{\|\Delta_r(|S_0u|^2)\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}} + \|\Delta_r(\sum_{l\geq r}\Delta_l u S_l\bar{u})\|_{L_x^{\frac{10}{7}} L_T^{\frac{10}{3}}}\right\}\\ &\lesssim \|S_0u\|_{L^2_x L^{10}_T}\|S_0u\|_{L^5_{x,T}} + \sum_{l\geq 1} 2^{\frac{3l}{5}}\|\Delta_l u\|_{L^5_x L_T^{\frac{10}{3}}}\|S_l u\|_{L^2_x L^\infty_T}\\ &\lesssim T^{\frac{3}{10}}\|S_0u\|_{L^2_x L^\infty_T}\|S_0u\|_{L^\infty_T L^5_x} + \sum_{l\geq 1} 2^{\frac{3l}{5}}\|\Delta_l u\|_{L^5_x L_T^{\frac{10}{3}}}\|S_l u\|_{L^2_x L^\infty_T}\\ &\lesssim T^{\frac{3}{10}} N_1(u)N_3(u) + N_3(u)N_2(u)^{\frac15}N_4(u)^{\frac45}, \end{aligned}$$

and therefore, by (66) we have

$$\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3,1}\|_{L^\infty_T L^2_x} \lesssim T^{\frac{13}{40}} N_1(u)N_3(u)N_4(u) + T^{\frac{1}{40}} N_3(u)N_2(u)^{\frac15}N_4(u)^{\frac95}. \tag{67}$$

To treat the second term on the RHS of (65), we use Lemma 2.2, Corollary 3.11, and the boundedness of S_{r-2} in $L^\infty_T L^2_x$ to get

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3,2}\|_{L^\infty_T L^2_x} &\lesssim \sum_{j\geq 1} 2^{\frac{j}{2}} \sum_{r\geq j} \|S_{r-2}uP\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2)\|_{L^1_x L^2_T}\\ &\lesssim \sum_{r\geq 0} 2^{\frac{r}{2}} \|S_{r-2}u\|_{L^2_x L^\infty_T}\|P\widetilde{D}_x^{\frac12}\Delta_{r-2}(|u|^2)\|_{L^2_{x,T}}\\ &\lesssim N_3(u)\sum_{r\geq 1} 2^r\|\Delta_r(|u|^2)\|_{L^2_{x,T}} \lesssim N_2(u)N_3(u)^2. \end{aligned} \tag{68}$$

Therefore, from (67) and (68) we obtain

$$\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_{1,3}\|_{L^\infty_T L^2_x} \lesssim T^{\frac{13}{40}} N_1(u)N_3(u)N_4(u) + T^{\frac{1}{40}} N_2(u)^{\frac15}N_3(u)N_4(u)^{\frac95} + N_2(u)N_3(u)^2. \tag{69}$$

Finally, we analyse $\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_2\|_{L_T^\infty L_x^2}$. First of all we use Minkowski inequality, the boundedness of the group $U(t)$ and Cauchy–Schwarz inequality to get

$$\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_2\|_{L_T^\infty L_x^2} \lesssim T^{\frac{1}{2}} \sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j(|u|^2 u)\|_{L^2_{x,T}}. \tag{70}$$

Using (16) and (18) with $f = u$ and $g = |u|^2$ and the boundedness of Δ_j we have

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j(|u|^2 u)\|_{L^2_{x,T}} &\lesssim \|S_0 u S_0(|u|^2)\|_{L^2_{x,T}} + \sum_{j\geq 1}\sum_{r\geq j} 2^{\frac{j}{2}}\|\Delta_{r-2} u S_{r-2}(|u|^2)\|_{L^2_{x,T}}\\ &\quad + \sum_{j\geq 1}\sum_{r\geq j} 2^{\frac{j}{2}}\|\Delta_{r-2}(|u|^2) S_{r-3} u\|_{L^2_{x,T}} =: I_{2,1} + I_{2,2} + I_{2,3}.\end{aligned}$$

In order to estimate $I_{2,1} + I_{2,2}$ we use Holder inequality, Sobolev embedding and (20) to get

$$\begin{aligned} I_{2,1} + I_{2,2} &\leq \||u|^2\|_{L^4_{x,T}}\|S_0 u\|_{L^4_{x,T}} + \sum_{j\geq 1}\sum_{r\geq j} 2^{\frac{j}{2}}\|\Delta_{r-2} u\|_{L^4_{x,T}}\|S_{r-2}(|u|^2)\|_{L^4_{x,T}}\\ &\lesssim T^{\frac{1}{4}}\|u\|^2_{L_T^\infty L_x^8} N_4(u) + T^{\frac{1}{4}}\|u\|^2_{L_T^\infty L_x^8}\sum_{r\geq 0} 2^{\frac{r}{2}}\|\Delta_r u\|_{L^4_{x,T}}\\ &\lesssim T^{\frac{1}{4}} N_1(u)^2 N_4(u). \end{aligned} \tag{71}$$

Next, we estimate $I_{2,3}$. We use once again Holder inequality, Sobolev embedding and (20) to find

$$I_{2,3} \lesssim T^{\frac{1}{6}}\|u\|_{L_T^\infty L_x^8}\sum_{r\geq 0} 2^{\frac{r}{2}}\|\Delta_r(|u|^2)\|_{L^3_{x,T}} \lesssim T^{\frac{1}{6}} N_1(u)\sum_{r\geq 0} 2^{\frac{r}{2}}\|\Delta_r(|u|^2)\|_{L^3_{x,T}}.$$

Now using (16) and (18) with $f = u$ and $g = \bar{u}$, Holder inequality and Sobolev embedding we get

$$\begin{aligned} I_{2,3} &\lesssim T^{\frac{1}{6}} N_1(u)\sum_{r\geq 0} 2^{\frac{r}{2}}\|\Delta_r(|u|^2)\|_{L^3_{x,T}}\\ &\lesssim T^{\frac{1}{6}} N_1(u)\{\|S_0 u\|_{L^{12}_{x,T}}\|S_0 u\|_{L^4_{x,T}} + \|\bar{u}\|_{L^{12}_{x,T}}\sum_{r\geq 0} 2^{\frac{r}{2}}\|\Delta_r u\|_{L^4_{x,T}}\}\\ &\lesssim T^{\frac{1}{4}} N_1(u)^2 N_4(u). \end{aligned} \tag{72}$$

From (70), (71) and (72) we conclude that

$$\sum_{j\geq 1} 2^{\frac{j}{2}}\|\Delta_j I_2\|_{L_T^\infty L_x^2} \lesssim T^{\frac{3}{4}} N_1(u)^2 N_4(u). \tag{73}$$

Gathering estimates (55), (57), (63), (67), (69), (73), we arrive to the result. □

In the next proposition we estimate the remaining norms.

Proposition 4.2. *Let $0 < T \leq 1$. Then*

$$\begin{aligned} N_m(I) &\lesssim T N_1(u)^3 + T^{\frac{3}{4}} N_1(u)^2 N_4(u) + T^{\frac{1}{2}} N_1(u) N_2(u) N_3(u) + T^{\frac{13}{40}} N_1(u) N_3(u) N_4(u)\\ &\quad + T^{\frac{1}{40}} N_2(u)^{\frac{1}{5}} N_3(u) N_4(u)^{\frac{9}{5}} + N_3(u)^2 N_2(u), \end{aligned} \tag{74}$$

for $m = 1, 2, 3, 4$.

Proof. The estimate for $N_1(I)$ is in Lemma 4.1.

Given $(p,q)\in\{(\infty,2),\,(2,\infty),\,(4,4)\}$ we have from Lemmas 3.3 and 3.8 that

$$\|U(t)S_0u_0\|_{L^p_xL^q_T}\lesssim T^{\frac{1}{q}}\|S_0u_0\|_{L^2}. \tag{75}$$

Then by Lemma 3.4 with $(p_1,q_1)=(p,q)$ and $(p_2,q_2)=(4,4)$ we get

$$\|S_0I_1\|_{L^p_xL^q_T}\lesssim T^{\frac{1}{q}}\|S_0(uP\partial_x(|u|^2))\|_{L^{\frac{4}{3}}_{x,T}}. \tag{76}$$

Therefore by (53) we find

$$\|S_0I_1\|_{L^p_xL^q_T}\lesssim T^{\frac{1}{2}}N_1(u)N_2(u)N_3(u). \tag{77}$$

Let us now estimate $\sum_{j\geq 1}2^{\frac{2j}{q}}\|\Delta_jI_1\|_{L^p_xL^q_T}$, for $(p,q)\in\{(\infty,2),\,(2,\infty),\,(4,4)\}$. As we did in (56), using (47) and (48) we write

$$\begin{aligned}\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_jI_1\|_{L^p_xL^q_T}\lesssim&\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_j\int_0^tU(t-s)S_0uS_0P\partial_x(|u|^2)ds\|_{L^p_xL^q_T}\\&+\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_j\int_0^tU(t-s)\sum_{r\geq j}\Delta_{r-2}uS_{r-3}P\partial_x(|u|^2)ds\|_{L^p_xL^q_T}\\&+\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_j\int_0^tU(t-s)\sum_{r\geq j}S_{r-2}u\Delta_{r-2}P\partial_x(|u|^2)ds\|_{L^p_xL^q_T}\\:=&\sum_{m=1}^{3}\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_jI_{1,m}\|_{L^p_xL^q_T}.\end{aligned} \tag{78}$$

From Lemmas 3.7 and 3.9 we infer

$$\|U(t)\Delta_ju_0\|_{L^p_xL^q_T}\lesssim 2^{\frac{j}{2}}\|\Delta_ju_0\|_{L^2}, \tag{79}$$

$$\|U(t)\Delta_ju_0\|_{L^4_{x,T}}\lesssim T^{\frac{1}{4}}\|\Delta_ju_0\|_{L^2}. \tag{80}$$

Then from Lemma 3.10, remark in (17) and the boundedness of $S_0P\partial_x$ in $L^{\frac{4}{3}}$ we deduce that

$$\begin{aligned}\sum_{j\geq1}2^{\frac{2}{q}j}\|\Delta_jI_{1,1}\|_{L^p_xL^q_T}&\lesssim T^{\frac{1}{4}}\|S_0uS_0P\partial_x(|u|^2)\|_{L^{\frac{4}{3}}_{x,T}}\\&\lesssim T\|S_0u\|_{L^\infty_TL^4_x}\|S_0P\partial_x(|u|^2)\|_{L^\infty_TL^2_x}\lesssim TN_1(u)^3.\end{aligned} \tag{81}$$

Employing Lemmas 3.7 and 3.9 and choosing $(\alpha_2,p_2,q_2)=(\frac{2}{5},20,\frac{20}{9})$ in Lemma 3.10 (by a similar argument as in (59)) we find

$$\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_jI_{1,2}\|_{L^p_xL^q_T}\lesssim T^{\frac{1}{40}}\sum_{j\geq1}2^{\frac{j}{10}}\sum_{r\geq j}\|\Delta_{r-2}uS_{r-3}P\partial_x(|u|^2)\|_{L^{\frac{20}{19}}_xL^{\frac{20}{11}}_T}. \tag{82}$$

Therefore, combining the above estimate with (60) and (62) we conclude that

$$\begin{aligned}\sum_{j\geq1}2^{\frac{2j}{q}}\|\Delta_jI_{1,2}\|_{L^p_xL^q_T}\lesssim&T^{\frac{1}{40}}\|D_x^{1/2}u\|_{L^4_{x,T}}\|u\|_{L^2_xL^\infty_T}\sum_{r\geq1}2^{\frac{3r}{5}}\|\Delta_ru\|_{L^5_xL^{\frac{10}{3}}_T}\\\lesssim&T^{\frac{1}{40}}N_2(u)^{\frac{1}{5}}N_4(u)^{\frac{9}{5}}N_3(u).\end{aligned} \tag{83}$$

In order to estimate $\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3}\|_{L^p_x L^q_T}$, we first split it as in (65):

$$\begin{aligned}\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3}\|_{L^p_x L^q_T} &\lesssim \sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j \int_0^t U(t-s) \sum_{r\geq j} [D_x^{\frac12}, S_{r-2}u] P\widetilde{D}_x^{\frac12} \Delta_{r-2}(|u|^2) ds\|_{L^p_x L^q_T} \\ &+ \sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j \int_0^t U(t-s) \sum_{r\geq j} D_x^{\frac12} (S_{r-2}uP\widetilde{D}_x^{\frac12} \Delta_{r-2}(|u|^2)) ds\|_{L^p_x L^q_T} \\ &:= \sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,1}\|_{L^p_x L^q_T} + \sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,2}\|_{L^p_x L^q_T}. \end{aligned} \tag{84}$$

From Lemmas 3.7 and 3.9 combined with Lemma 3.10 with $(\alpha_2, p_2, q_2) = (\frac{2}{5}, 20, \frac{20}{9})$ (see estimate (59)), we deduce that

$$\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,1}\|_{L^p_x L^q_T} \lesssim T^{\frac{1}{40}} \sum_{j\geq 1} 2^{\frac{j}{10}} \sum_{r\geq j} \|[D_x^{\frac12}, S_{r-2}u] P\widetilde{D}_x^{\frac12} \Delta_{r-2}(|u|^2)\|_{L_x^{\frac{20}{19}} L_T^{\frac{20}{11}}}. \tag{85}$$

The expression on the RHS of (85) was already treated in (66)–(67). Therefore

$$\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,1}\|_{L^p_x L^q_T} \lesssim T^{\frac{13}{40}} N_1(u) N_3(u) N_4(u) + T^{\frac{1}{40}} N_3(u) N_2(u)^{\frac15} N_4(u)^{\frac95}. \tag{86}$$

To finish, using Lemmas 2.2, 3.12 (with $(p,q) = (\infty, 2)$), Lemma 3.10 (with $(p,q) = (2,\infty)$ or $(p,q) = (4,4)$) and the boundedness of S_{r-2} in $L^p_x L^q_T$, we get

$$\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,2}\|_{L^p_x L^q_T} \lesssim \sum_{j\geq 1} 2^{\frac{j}{2}} \sum_{r\geq j} \|S_{r-2}uP\widetilde{D}_x^{\frac12} \Delta_{r-2}(|u|^2)\|_{L^1_x L^2_T}. \tag{87}$$

And then, from the estimate in (68) we see that

$$\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3,2}\|_{L^p_x L^q_T} \lesssim N_2(u) N_3(u)^2. \tag{88}$$

From the above estimate and (86) we deduce that

$$\sum_{j\geq 1} 2^{\frac{2j}{q}} \|\Delta_j I_{1,3}\|_{L^p_x L^q_T} \lesssim T^{\frac{13}{40}} N_1(u) N_3(u) N_4(u) + T^{\frac{1}{40}} N_2(u)^{\frac15} N_3(u) N_4(u)^{\frac95} + N_2(u) N_3(u)^2. \tag{89}$$

As in (73), a straightforward calculation shows that

$$N_m(I_2) \lesssim T^{\frac34} N_1(u)^2 N_4(u), \qquad m = 2,3,4. \tag{90}$$

Thus the proposition is proven. □

4.2. Proof of Theorem 1.1

The goal is to establish local well-posedness for the Cauchy problem associated with (1) in the Besov space $B_2^{\frac12,1}(\mathbb{R})$.

Given u_0 in $B_2^{\frac12,1}(\mathbb{R})$, we look for a (unique) solution in the space X_T $(0 < T \leq 1)$ defined in (27). Let

$$\Phi(u)(t) = U(t)u_0 + I,$$

where I is defined in the RHS of (46). Hence, by gathering Propositions 3.13 and 4.2 we see that

$$\begin{aligned}\|\Phi(u)\|_{X_T} \lesssim & \|u_0\|_{B_2^{\frac{1}{2},1}} + TN_1(u)^3 + T^{\frac{3}{4}}N_1(u)^2N_4(u) + T^{\frac{1}{2}}N_1(u)N_2(u)N_3(u) \\ & + T^{\frac{13}{40}}N_1(u)N_3(u)N_4(u) + T^{\frac{1}{40}}N_2(u)^{\frac{1}{5}}N_3(u)N_4(u)^{\frac{9}{5}} + N_3(u)^2N_2(u),\end{aligned}$$

and thus, since $0 < T \leq 1$ we get,

$$\begin{aligned}\|\Phi(u)\|_{X_T} \leq & C\|u_0\|_{B_2^{\frac{1}{2},1}} + C(T + T^{\frac{3}{4}} + T^{\frac{1}{2}} + T^{\frac{13}{40}} + T^{\frac{1}{40}} + 1)\|u\|_{X_T}^3 \\ \leq & C\|u_0\|_{B_2^{\frac{1}{2},1}} + C\|u\|_{X_T}^3. \end{aligned} \tag{91}$$

Consider $\delta = (4C)^{-\frac{3}{2}}$ and $u_0 \in B_2^{\frac{1}{2},1}(\mathbb{R})$ satisfying $\|u_0\|_{B_2^{\frac{1}{2},1}} \leq \delta$. Choosing $a = 4C\|u_0\|_{B_2^{\frac{1}{2},1}}$, we have from (91) that

$$\|\Phi(u)\|_{X_T} \leq \frac{a}{4} + Ca^3 \leq a,$$

for all $u \in X_T^a$. That gives us $\Phi(X_T^a) \subseteq X_T^a$.

An analogous approach leads to the estimate

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq C(\|u\|_{X_T}^2 + \|v\|_{X_T}^2)\|u - v\|_{X_T}.$$

Therefore,

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq \frac{1}{2}\|u - v\|_{X_T},$$

for all $u, v \in X_T^a$. The remainder of the proof follows from a standard argument.

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References

[1] L. Abdelouhab, J.L. Bona, M. Felland, J.C. Saut, Nonlocal models for nonlinear, dispersives waves, Physica D 40 (1989) 360–392.

[2] J. Angulo, R.P. Moura, Ill-posedness and the nonexistence of standing-waves solutions for the nonlocal nonlinear Schrödinger equation, Differential Integral Equations 20 (2007) 1107–1130.

[3] J. Bergh, J. Lofstrom, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[4] M. Christ, A. Kiselev, Maximal functions associated to filtrations, J. Funct. Anal. 179 (2001) 409–425.

[5] C.E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J. 40 (1991) 33–69.

[6] C.E. Kenig, G. Ponce, L. Vega, Well-posedness of the initial value problem for the Korteweg–de Vries equation, J. Amer. Math. Soc. 4 (1991) 323–347.

[7] C.E. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré Anal. Non Linéaire 10 (1993) 255–288.

[8] Y. Matsuno, Linear stability of multiple dark solitary wave solutions of a nonlocal nonlinear Schrödinger equation for envelope waves, Phys. Lett. A 285 (2001) 286–292.

[9] Y. Matsuno, Multiperiodic and multisoliton solutions of a nonlocal nonlinear Schrödinger equation for envelope waves, Phys. Lett. A 278 (2001) 53–58.

[10] Y. Matsuno, Calogero-Moser-Sutherland dynamical systems associated with nonlocal nonlinear Schrödinger equation for envelope waves, J. Phys. Soc. Japan 71 (2002) 1415–1418.

[11] Y. Matsuno, Dark soliton generation for the intermediate nonlinear Schrödinger equation, J. Math. Phys. 43 (2002) 984–1007.

[12] Y. Matsuno, Exactly solvable eigenvalue problem for a nonlocal nonlinear Schrödinger equation, Inverse Probl. 18 (2002) 1101–1125.

[13] Y. Matsuno, Asymptotic solutions of the nonlocal nonlinear Schrödinger equation in the limit of small dispersion, Phys. Lett. A 309 (2003) 83–89.
[14] Y. Matsuno, A Cauchy problem of the nonlocal nonlinear Schrödinger equation, Inverse Probl. 20 (2004) 437–445.
[15] L. Molinet, F. Ribaud, Well-posedness results for the generalized Benjamin–Ono equation with arbitrary initial data, Int. Math. Res. Not. 70 (2004) 3757–3795.
[16] L. Molinet, F. Ribaud, Well-posedness results for the generalized Benjamin–Ono equation with small initial data, J. Math. Pures Appl. 9 (7) (2004) 277–311, 83.
[17] R.P. Moura, Well-Posedness for the nonlocal nonlinear Schrödinger Equation, J. Math. Anal. Appl. 326 (2007) 1254–1267.
[18] R.P. Moura, D. Pilod, Local Well-Posedness for the nonlocal nonlinear Schrödinger Equation below the energy space, Adv. Differential Equations 15 (9–10) (2010) 925–952.
[19] T. Ozawa, On the nonlinear Schrödinger equations of derivative type, Indiana Univ. Math. J. 45 (1996) 137–163.
[20] D.E. Pelinovsky, Intermediate nonlinear Schrödinger equation for internal waves in a fluid of finite depth, Phys. Lett. A 197 (1995) 401–406.
[21] D.E. Pelinovsky, R.H.J. Grimshaw, A spectral transform for the intermediate nonlinear Schrödinger equation, J. Math. Phys. 36 (1995) 4203–4219.
[22] D.E. Pelinovsky, R.H.J. Grimshaw, Nonlocal models for envelope waves in a stratified fluid, Stud. Appl. Math. 97 (1996) 369–391.
[23] D. Pilod, On the cauchy Problem for the higher-order nonlinear dispersive equation, J. Differential Equations 245 (2008) 2055–2077.
[24] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, Adv. Differential Equations 4 (1999) 561–580.